Capital Markets, Output, and the Demand for Inputs Under Uncertainty

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1. Introduction

The Sharpe (1964)-Litter (1965)-Mossin (1966) capital-asset-pricing-model has motivated considerable contributions to the theory of finance and to the empirical verification of financial propositions. This model provides a capital market equilibrium approach to the valuation of risky claims and its various applications include studies of corporate financial policies, investment (capital budgeting) decisions, and examinations of the efficiency of capital markets. The above studies culminate in a financial theory of the firm which sheds light on the valuation of securities under uncertainty while neglecting the classical problems of the firm, concerning the choice of factor-proportions and the determination of its output. Moreover, the financial literature is curiously silent with respect to the equilibrium of the industry in terms of price-output and inputs adjustments and the associated entry (exit) of firms.

Meanwhile, the incorporation of uncertainty into the neoclassical theory of the firm took place within a class of "entrepreneurial* models in which the firm is assumed to maximize the expected utility of random profits. These models focus on production under uncertainty, where the firm is a participant in the product and factor markets, without any reference to the role of financial markets as a mechanism for the valuation of uncertain profit claims and allocation of resources. This class of models indicates that production decisions are sensitive to the preferences of the entrepreneur and that, consequently, the results of the deterministic theory of the firm do not in general prevail. Since capital markets are absent within this class of models, one is still left with the task of separating the impacts of uncertainty and risk aversion, personal taste, and those which are due to the assumed incomplete market setting where risky prospects are valued subjectively through managerial preferences.

This paper integrates the financial literature, of valuation of securities under uncertainty, with the standard neoclassical theory of the firm dealing with a production plan and with the equilibrium adjustments of the industry in terms of inputs, price-output determinants, and demand for inputs under uncertainty.  

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1Examples of the application of the capital-asset-pricing-model include: Pliska (1970), Stigum (1972), Fama (1972), Jensen and Meckling (1972), Rubinstein (1973), and Mullin and Sakamoto (1973).  
2The literature of the utility-maximizing firm under uncertainty received considerable impetus with the publication of Sandmo (1973) and Letzr (1972). Further studies of this subject include McCall (1965) and Burstein (1970). Extensions of this approach to the study of the demand for inputs under uncertainty include; Baer and Wilson (1974), Hartman (1975), Holtzhausen (1976) and Epstein (1976).
mation, and the associated entry (exit) of firms. In section II we apply the capital-asset-price-model to a firm which operates under demand and factor-cost uncertainty. Then, it is shown that the firm, by choices over various bundles of inputs, generates combinations of expected profits and risk which are valued in the financial market consistently with the requirements of capital market equilibrium. In this setting, firms serve the interest of their shareholders by adherence to the maximum market value rule. Furthermore, examination of the optimizing behavior of the firm under uncertainty and risk aversion reveals that the standard results of the deterministic theory obtain with capital market established certainty-equivalents in place of the certain market prices. But these results differ from the risk-neutrality results, and the differences are fully explained by risk aversion considerations. Thus, for example, we show that a firm facing a negative systematic uncertainty of output price and risk aversion produces a greater output and employs greater (smaller) amounts of normal (inferior) inputs than the risk neutral firm.

II. Risk, Expected Returns and the Production Decision

The mean-standard deviation approach to choices over risky claims yields the capital-asset-price-model as a characterization of capital market equilibrium. Within this framework risky claims are valued in accordance with the following formula:

\[
V_{ik} = \frac{1}{1 + \tau} E[V_R] - \lambda \text{Cov}(R, \hat{M})/\sigma_\hat{M},
\]

where

- \( R_t \) = the end-of-period random cash flow claimed by the owners (investors) of firm \( k \);
- \( M_t \) = the end-of-period market portfolio; \( M_t \) is the market portfolio;
- \( E(\cdot) \) = expectation operator;
- \( \text{Cov}(\cdot, \cdot) \) = covariance operator;
- \( \sigma_M \) = the standard deviation of \( M_t \);
- \( \tau = (1 + r) \) = one plus the one-period riskless rate of interest;
- \( V_{ik} \) = the present total value of all firms, i.e., the market value of firm \( M_t \) at times 0;
- \( \lambda = E[M_t - \hat{M}] / \sigma_M \) = the coefficient of aggregate risk aversion of the market which is commonly referred to as the market price of risk;
- \( \hat{M} \) = the mean present market value of firm \( k \).

Now, in order to ascertain the consistency of our production decisions with the requirements of capital market equilibrium and owners' wealth maximization, we modify equation (1) to the needs of a firm which operates under uncertain demand and faces uncertain input prices. According to equation (1) the net present value of the firm is

\[
V_{ik} = \frac{1}{\tau} \left[ E[(R_t - \hat{R}_t)K] - \lambda \text{Cov}(R_t, \hat{M})/\sigma_{\hat{M}} \right],
\]

where \( K \) represents the number of units of capital employed by the firm and \( \tau \) stands for the present (certain) acquisition price per unit of capital input. Utilizing the equalities \( \Pi = \text{Cov}(\Pi_t, \hat{M})/\sigma_{\hat{M}} \) and \( \Pi_t = \text{Cov}(\Pi_t, \hat{M})/\sigma_{\hat{M}} \) and \( \Pi_t \) = random variables distributed independently of the levels of \( K \) and \( \sigma_{\hat{M}} \) for \( i = 1, 2, \ldots, n \), where:

\[
\Pi_t = \text{the selling price of a unit of output.}
\]

First, note that the capital-asset-price-model is derived on the assumption that investors evaluate random returns in accordance with the mean and variance only. Thus, our equation (2) embodies the assumption that either \( R_t \) and \( \hat{M} \) are normally distributed or that investors have quadratic utility functions. Second, price-taking behavior in the financial market assumes that the firm has no perceptible effect on the mean and variance of \( \hat{M} \) and that it is a market parameter which is unaffected by the decisions of the firm. This definition of price-taking is equivalent to Fama's "reaction principle" of competitive behavior (1972).

Let \( q = w_t(\hat{M}), K \) be the twice differentiable production function of \( n + 1 \) current inputs and one (homogeneous) capital input:

\[
w_t(\hat{M}, K) = \text{input price times quantity of input}
\]

\( i \)-payment to factor \( i \);

\( b = \) the rate of economic depreciation (replacement) of capital assets which is assumed to be proportional to the capital stock; and

\( z_t(\hat{M}, K) = \) the end-of-period liquidation value of the stock of capital.

Under the above specifications the first-order condition for the maximization of \( \Pi_t \) with respect to \( x_t \) are:

\[
\frac{\partial \Pi_t}{\partial x_t} = \frac{E[\Pi_t]}{\lambda} - \lambda \text{Cov}(x_t, \hat{M})/\sigma_{\hat{M}} = 0,
\]

where \( \Pi_t = \theta_t(\hat{M})), K) \) is the random marginal profit contribution of factor \( x_t, x_t = K \) with the factor wage \( w_t(\hat{M}) = (\tau + z_t(\hat{M}), K) \) and \( \text{Cov}(\Pi_t, \hat{M})/\sigma_{\hat{M}} = f(\hat{M}, K)/\sigma_{\hat{M}} \) and \( \text{Cov}(\Pi_t, \hat{M})/\sigma_{\hat{M}} = f(\hat{M}, K)/\sigma_{\hat{M}} \), where \( p_t = \text{the average profit function of firm } i \) and \( \phi_t = \text{the average profit function of firm } i \) and \( \phi_t = \text{the average profit function of firm } i \) and \( \phi_t = \text{the average profit function of firm } i \).
spectively, the systematic risk of \( P \) and the systematic risk of \( w \). The first-order conditions of equations (4) state that the optimal inputs of a competitive firm under uncertainty and risk aversion are determined by the equality of the certainty-equivalent value of the marginal product of each input \( \left( P_f = E(P_f) - b_{w,f}f_w \right) \) to its certainty-equivalent input price \( \left( p_{w,f} = E(w_{f}) - b_{w}f_{w} \right) \). It is easy to verify that

\[
\frac{d\tilde{P}_i}{d\tilde{w}_i} = \frac{E(P_f) - b_{w,f}f_w}{p_{w,f}} = \frac{E(P_f)}{p_{w,f}}
\]

for \( i = 1, 2, \ldots, n \).

A sufficient condition for \( \tilde{P} \) to have a local maximum is that, together with equations (4), the matrix \( \left( \frac{d^2\tilde{P}_i}{d\tilde{w}_i d\tilde{w}_j} \right) \) be negative definite, which we assume to be the case. By equations (5) this implies that the matrix of cross-partial derivatives of the production function, \( \left( f_{2,f} \right) \), is negative definite, i.e., locally strictly concave. Hence the second-order conditions for the competitive firm under uncertainty and risk aversion are identical to those obtained in the standard profit-maximization models under certainty. Further examination of equations (4) shows that the optimal firm's product is not affected by fixed cost and that the supply curve of the firm is positively sloped. Thus we conclude that Sandro's results, where output is affected by fixed cost and where the supply curve of a competitive firm could be negatively sloped, do not follow from uncertainty and risk aversion alone; they are the consequences of the incomplete market setting where uncertain profits are valued subjectively.

II. Marginal Impacts of Uncertainty and Risk Aversion

We are interested in the effects of the optimal factor demands and output of changes in the marginal uncertainty. The effect of an increase in the marginal uncertainty of the firm's inputs on the optimal factor demands and output is determined by the certainty-equivalent prices of the firm's inputs. The certainty-equivalent price of an input is defined as the expected value of the input minus the expected value of the input's uncertainty. The certainty-equivalent price of an input is determined by the certainty-equivalent value of the marginal product of the input minus the expected value of the input's uncertainty. The certainty-equivalent value of the marginal product of an input is the expected value of the marginal product of the input minus the expected value of the input's uncertainty. The certainty-equivalent value of the marginal product of an input is determined by the certainty-equivalent value of the marginal product of the input minus the expected value of the input's uncertainty.
in uncertainty or risk aversion. To investigate these effects, we note that the assumption of the existence of an interior maximum allows us to solve equations (4) for the optimal levels of inputs in terms of the certainty-equivalent prices, i.e., \( x^* = x(P; \bar{w}, \ldots, \bar{w}), i = 1, 2, \ldots, n. \)

Let us first investigate the effects of changes in the systematic uncertainty of input prices. Differentiating (4) with respect to \( \theta \), \( \bar{w} \), \( \bar{w} \), yields the following system of equations:

\[
\left( \frac{\partial P_j}{\partial \theta} \right) \left( \frac{\partial x^*}{\partial \theta} \right) = \left( \frac{\partial x^*}{\partial \theta} \right), \quad (7)
\]

where \( x^* \) and \( \bar{w} \) are column vectors of inputs and their respective certainty-equivalent input prices, so that \( \frac{\partial x^*}{\partial \theta} = (\frac{\partial x^*_1}{\partial \theta}, \ldots, \frac{\partial x^*_n}{\partial \theta}) \) and \( \frac{\partial \bar{w}}{\partial \theta} = (\frac{\partial \bar{w}_1}{\partial \theta}, \ldots, \frac{\partial \bar{w}_n}{\partial \theta}) \). Solving (7) for \( \frac{\partial x^*}{\partial \theta} \) by means of Cramer's rule, we obtain

\[
\frac{\partial x^*}{\partial \theta} = -\frac{H_j}{H} \left( \frac{\partial \bar{w}_j}{\partial \theta} \right), \quad i = 1, 2, \ldots, n, \quad (8)
\]

where \( H_j \) is the determinant of the \( n \times n \) matrix \( (\partial P_j/\partial \theta) \), and \( H_j \) is the cofactor of the element in row \( j \) and column \( i \) of this matrix. Since the sign of the off-diagonal cofactor is unknown, we see that \( \frac{\partial x^*}{\partial \theta} \) for \( i \neq j \) is indeterminate. However, by the maximization hypothesis \( H \) and \( H_j \) have opposite signs and for \( i = j \) we have \( \frac{\partial x^*}{\partial \theta} > 0 \) by virtue of (8). Thus, we have shown that under uncertainty and risk aversion the optimal amount of an input employed by the firm increases with an increase in the uncertainty of its price. While this result may appear to be counter-intuitive, actually, its validity is easily verified by noting that for \( i = j \), equation (8) is

\[\text{x}^* \text{ rotates in the x-y plane with respect to } \theta. \]

\[\text{The symbol } \ast \text{ denotes optimal (wealth-maximizing) values. Note that } x^* \text{ and } x^* \text{ are deterministic values which depend on the certainty-equivalent prices only.}\]

\[\text{of a change in the price uncertainty of the } i^\text{th} \text{ input on the demand for the } j^\text{th} \text{ input.}^{14}\]

Now we turn to the effect of a change in the uncertainty of the price of an input on the output of the firm. Substituting the solution of (4) into the production function, we obtain the indirect production function \( q^* = f(x^*, \ldots, x^*, \bar{w}_j) \) where the optimal output is now, via \( x^* = x(P; \bar{w}_j, \ldots, \bar{w}_n) \), for \( i = 1, 2, \ldots, n, \) a function of market parameters. Then

\[
\frac{\partial q^*}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial q^*}{\partial \theta} = \frac{\partial q^*}{\partial \theta} \quad \text{for } i = 1, 2, \ldots, n, \quad \text{(9)}
\]

The above equation shows that under uncertainty and risk aversion the optimal amount of an input factor employed by the firm decreases (increases) with an increase in the uncertainty of output price if the input factor is normal (inferior).\(^{15}\)

Second, utilizing the symmetry of the matrix \( (\frac{\partial P_j}{\partial \theta}) \) one obtains \( H_j = H_j \) and, therefore, equation (8) shows that the reciprocity relation \( \frac{\partial x^*}{\partial \theta} \) \( = \frac{\partial x^*}{\partial \theta} \) holds for \( i = 1, 2, \ldots, n, \) i.e., the impact of a change in the price uncertainty of the \( j^\text{th} \) input on the demand for the \( i^\text{th} \) input is equal to the impact

\[\text{We are, actually, interested in the impact of a change in } \theta \text{ on the market value of the firm. But equation (2) shows } \bar{w}_j = \bar{w}_j/\theta \text{ and, therefore, sign } \frac{\partial \bar{w}_j}{\partial \theta} = \text{ sign } \frac{\partial \bar{w}_j}{\partial \theta}.\]

\[\text{This result can be obtained directly from the invertibility of the cross-product to the order of differentiation, i.e., } \frac{\partial x^*}{\partial \theta} \frac{\partial x^*}{\partial \theta} = \text{ sign } \frac{\partial \bar{w}_j}{\partial \theta}.\]

\[\text{See Ferguson (1968), p. 143.}\]
equation (14) states that under uncertainty and risk aversion the optimal output of the firm decreases with an increase in the uncertainty of output price. An immediate implication of this result is that the market value of the firm is a convex decreasing function of the uncertainty of output price. That is,

$$\frac{\Delta M^*}{\delta \sigma} = \sum_{i=1}^{n} (\delta P_i - \theta_i) \frac{\Delta x_{i^*}}{\sigma} = -\lambda q^* \sigma < 0 \quad (15)$$

and

$$\frac{\Delta^2 \hat{M}}{\delta \sigma^2} = -\lambda q^* > 0. \quad (16)$$

A comparison of equations (9) and (15) reveals that, in general, cost uncertainty affects the value of the firm in the opposite direction of revenue uncertainty. This is easily explained by examining the expression

$$\sigma_{\Delta \hat{M}/\Delta \sigma} = \text{Corr}(\Delta \hat{M}/\Delta \sigma, \sigma) = \text{Corr}(\Delta \hat{M} - \sum_{i=1}^{n} x_i \Phi_i (\sigma) ) / \sigma$$

where \(\Phi_i (\sigma) = \Phi_i (\sigma) - \theta_i \sigma < 0. \) The capital-assets-pricing model shows a negative relation between the market value of the firm and the systematic risk of its profit. But an increase in cost (revenue) uncertainty reduces (increases) the systematic uncertainty of profit and, consequently, the market value of the firm rises (falls). Alternatively, an increase of \(\theta_i \) reduces the certainty-equivalent factor cost (output price) which causes a rise (fall) in the value of the firm.

We further note that the conventional capital-assets-pricing-model of financial exchange relations leaves linear the value of the firm to its systematic risk, while our more general production and exchange framework shows that the functional relationship is actually convex. Equations (15) and (16) show, clearly, that this convexity is due to the adjustments in the inputs and output which are presumed to be fixed in the traditional pure exchange framework.

Equations (11) and (13) reveal, instantly, that \(\Delta q^*/\Delta \sigma < 0 = -\Delta x_{i^*}/\sigma \). Furthermore, equation (14) shows that \(\Delta x_{i^*}/\Delta \sigma < 0 \) while, by equation (13), the sign of \(\Delta x_{i^*}/\Delta \sigma \) is negative (positive) if \(x_i \) is normal (inferior). These two equations suggest that a demand uncertainty induced change in output will cause a change in the same (opposite) direction of an input if the input is normal (inferior). Two additional observations, based on equations (13) and (11), respectively, are that for a firm with a homothetic production function: (1) the employment of all input factors falls with an increase in the uncertainty of output price, and (2) the optimal output increases with an increase in the uncertainty of an input price. These results follow from the well-known fact that homotheticity rules out inferior inputs. Finally, we turn to investigate the consequences of changes in risk aversion. Lintner (1950) and Rubinstein (1973) have shown that the market price of risk, \(\lambda \), is a composite index of the risk aversion coefficients of all consumers (investors). It was shown that \(\lambda \) may change due to changes in variables like risk preferences of investors, market size, or the level and distribution of aggregate wealth. Differentiation of (4) with respect to \(\lambda \) results in the following system of equations:

$$\left(\frac{\partial P_i}{\partial \lambda} \right) \frac{\Delta x_{i^*}}{\partial \lambda} = \left( -\frac{\partial P_i}{\partial \lambda} f_i - \frac{\partial f_i}{\partial \lambda} \right), \quad (17)$$

where \(\Delta x_{i^*}/\Delta \lambda = (\Delta x_{i^*}/\Delta \lambda, \ldots, \Delta x_{i^*}/\Delta \lambda)\) and \(\left(\Delta f_i/\Delta \lambda \right) = (\theta_i - \theta_i, \ldots, \theta_i - \theta_i)\). Solving for \(\Delta x_{i^*}/\Delta \lambda \), one obtains

$$\Delta x_{i^*}/\Delta \lambda = \sum_{i=1}^{n} (\theta_i - \theta_i) H_i \left( \frac{\partial H_i}{\partial \lambda} \right) \quad (18)$$

It is clear that at this level of generality, no prediction as to the sign of \(\Delta x_{i^*}/\Delta \lambda \) is forthcoming. This ambiguity is not surprising since a shift in \(\lambda \) changes simultaneously the certainty-equivalent prices of all inputs and outputs. However, in the special case where uncertainty is associated only with the factor price \(\theta_i \), equation (18) specializes into \(\Delta x_{i^*}/\Delta \lambda = -\theta_i H_i/\lambda \geq 0 \) if \(\theta_i \geq 0 \). Similarly, in the special case of output price uncertainty, equation (18) becomes \(\Delta x_{i^*}/\Delta \lambda = \theta_i \left( \sum_{i=1}^{n} \left( f_i H_i/\lambda \right) \right) \) with the result that the employment of a normal input factor decreases (increases) with an increase in aggregate risk aversion if \(\theta_i \) is positive (negative). Of course, the effect is reversed for an inferior input. Further examination of the special case of output price uncertainty yields

$$\Delta x_{i^*}/\Delta \lambda = \sum_{i=1}^{n} f_i \frac{\Delta x_{i^*}}{\delta \lambda} = \sum_{i=1}^{n} f_i \left( \frac{\partial H_i}{\partial \lambda} \right) \quad (19)$$

Since the above double summation represents a negative definite quadratic form, it follows that under output price uncertainty and risk aversion the output of the firm decreases (increases) with an increase in aggregate risk aversion if the systemic uncertainty of output price is positive (negative).

**IV. The Firm in a Long-Run Equilibrium Under Uncertainty**

In this section we turn to analyze the responses of the firm to changes in uncertainty with the specific proviso that the firm is in a competitive long-run equilibrium. We adopt the Ferguson and Saving (1969) and Silberberg (1974a and 1974b) notion of a competitive long-run equilibrium, where output price adjusts freely in response to changes in the underlying parameters of the average cost curve. This approach overcomes the inconsistency of holding output price constant in the face of changing cost through changes of factor prices. This concept of a competitive long-run equilibrium is based on the following assumptions: (1) the industry is composed of a large number of identical firms so that the same technology is available to all firms, extant and potential; (2) the industry is characterized by free entry and exit, i.e., firms face zero entry costs and no adjustment costs; (3) all input factors are flexible and freely adjustable; and (4) the firm is a price taker in the markets for inputs and output as well as in the financial markets.

Under the above assumptions, entry and exit of firms assures that a competitive long-run equilibrium occurs at \(\hat{H} = E(\Pi) = 0 \). Every firm is characterized by a competitive long-run equilibrium.

$$\hat{P} = E(\hat{P}) = \hat{\theta}_i \hat{H} \quad (20)$$

Rearrangement of equation (20) as \(E(\hat{P}) - \hat{\theta}_i \hat{H} = 0 \) shows that, in a competitive long-run equilibrium under uncertainty and risk aversion, expected output price is greater (smaller) than expected average cost so that expected profit is positive (negative) if output price uncertainty is greater (smaller) than the uncertainty of average cost. Thus the essence of a competitive long-run equilibrium is the requirement of zero certainty-equivalent profit which, in contrast to the typical findings concerning the firm under uncertainty, imposes no restriction on the sign of expected profits.

Our long-run equilibrium requires that the certainty-equivalent output price be identical.
cally equal to the certainty-equivalent average cost and that \( P \) always adjusts to changes in the minimum \( AC \). Clearly, the wealth maximization policy of a firm under these conditions calls for the choice of that vector of inputs which minimizes \( AC \). The minimization of \( AC \) is, obviously, equivalent to the maximization of the following function

\[
L = \hat{AC}^* - \hat{AC} = \sum_{i=1}^{n} (w_i - \lambda \hat{y}_i) x_i / q_i,
\]

(21)

where \( \hat{AC}^* = \hat{AC}^*(\hat{\alpha}_1, \ldots, \hat{\alpha}_n) \) is the dual (indirect) minimum certainty-equivalent average cost as a function of the vector of parameters \( \hat{\alpha}_1, \ldots, \hat{\alpha}_n \). Therefore the function \( L = \hat{AC}^* - \hat{AC} \geq 0 \), by construction, a maximum value of zero when \( x^* = x_i \) for \( i = 1, 2, \ldots, n \).

Partial differentiation of \( L \) with respect to \( x_i \) yields the first order conditions:

\[
\frac{\partial L}{\partial x_i} = -\hat{AC}_i = \frac{1}{q_i} (w_i - \lambda \hat{y}_i) \hat{AC}_i = 0,
\]

(22)

and the second order conditions require that the following matrix be negative definite:

\[
\hat{AC}_{ij} = \frac{\partial^2 \hat{AC}}{\partial x_i \partial x_j} < 0,
\]

(23)

Conditions (22) show that the optimal amount of each input is determined by the equality of the certainty-equivalent value of the marginal product of each input to its certainty-equivalent input price.\(^2\)

The sufficient condition for a long-run equilibrium to exist in the uncertainty of input prices on the relative factor demand \( \hat{y}_i / \hat{y}_j \) is that, together with equation (22), the matrix of cross partials of the production function, \( \hat{AC}_{ij} \), is locally negative definite.

Now we pay attention to the impact of changes in the uncertainty of input prices on the relative factor demand \( \hat{y}_i / \hat{y}_j \). Differentiating (21) and setting \( \partial \hat{L} / \partial \hat{y}_i = \hat{AC}_i \hat{AC}_{ij} / \hat{AC}_{ii} \partial \hat{AC}_{ij} / \partial \hat{y}_j = 0 \), one obtains

\[
\frac{\partial \hat{AC}}{\partial \hat{y}_j} = -\lambda (\hat{\alpha}_i / \hat{\alpha}_j) < 0,
\]

(24)

\( j = 1, 2, \ldots, n \).

The second derivative of (21) with respect to \( \hat{y}_j \) is \( \partial^2 \hat{L} / \partial \hat{y}_j^2 = \hat{AC}_{ij} \hat{AC}_{ii} / \hat{AC}_{jj} \leq 0 \), where the negative sign follows from the fact that \( \hat{AC}^* / \hat{AC} \) is a first-order principal minor (diagonal element) of the negative semidefinite matrix of cross partials of \( L \).\(^2\)

Equation (25) states that in a competitive long-run equilibrium under uncertainty and risk aversion the relative factor demand increases with an increase in the factor's price uncertainty:

\[
\frac{\partial \hat{y}_i / \partial \hat{y}_j}{\partial \hat{y}_j} = \lambda (\hat{\alpha}_i / \hat{\alpha}_j) > 0,
\]

(25)

\( j = 1, 2, \ldots, n \).

Equation (25) states that in a competitive long-run equilibrium under uncertainty and risk aversion the relative factor demand increases with an increase in the factor's price uncertainty:

\[
\frac{\partial \hat{y}_i / \partial \hat{y}_j}{\partial \hat{y}_j} = \lambda (\hat{\alpha}_i / \hat{\alpha}_j) > 0,
\]

(25)

\( j = 1, 2, \ldots, n \).

In contrast to (25), equation (8) shows that in a short-run equilibrium \( \delta \hat{y}_i / \partial \hat{y}_j > 0 \) even when output decreases. An additional result concerning relative factor demand is richness with respect to changes in the uncertainty of input prices: \( \delta \hat{y}_i / \partial \hat{y}_j > 0 \) if \( \delta \hat{x}_i / \partial \hat{y}_j > 0 \), \( i = 1, 2, \ldots, n \). These equations follow from the symmetry of the matrix of cross partials of \( L \) (invariance with respect to the order of differentiation).

Next we examine the impact of a change in the uncertainty of input price on the output of the firm. In accordance with equations (22) and (23), the minimum certainty-equivalent average cost (point of \( MC(W_0, \ldots, W_n, \hat{\alpha}_i, \ldots, \hat{\alpha}_n) = AC(W_0, \ldots, W_n, \hat{\alpha}_i, \ldots, \hat{\alpha}_n) \)) is implicit differentiation with respect to \( \hat{y}_i \), yields \( \delta \hat{MC} / \partial \hat{y}_j = (\delta \hat{MC} / \partial \hat{y}_i) \delta \hat{y}_j / \partial \hat{y}_j + (\delta \hat{MC} / \partial \hat{y}_j) \delta \hat{y}_i / \partial \hat{y}_j = 0 \).

Expressions (26) and (27) show that, first, a sufficient condition for the long-run equilibrium of an input to rise with an increase in its price uncertainty is that the factor's production function is homogeneous of degree one, i.e., \( \delta \hat{y}_i / \partial \hat{y}_j > 0 \) if \( \hat{y}_j > 0 \). A second consequence of equations (26) and (27) is that, if the production function of the firm is homothetic, the long-run output remains constant (at the output price) with an increase in the uncertainty of the input's price.

This last statement follows from the fact that a long-run equilibrium for a firm with a homothetic production function requires \( \hat{y}_j > 0 \).
\[ \frac{\partial C}{\partial C} = 1 \text{ for } j = 1, 2, \ldots, n. \] Therefore, \( \frac{\partial \bar{q}}{\partial \bar{b}_0} = 0 \) by (27) and \( \frac{\partial \bar{q}}{\partial \bar{b}_0} > 0 \) by (26).

### V. Industry Equilibrium

Under Uncertainty

All the assumptions of section IV, concerning the structure of the industry and the long-run equilibrium behavior of competitive firms, are maintained in this section. Additionally, it is assumed that the certainty-equivalent factor prices are parameters in the industry and that the certainty-equivalent demand curve for the output of the industry is negatively sloped.

As a consequence of the assumptions of a purely competitive equilibrium, where all firms are identical and entry (exit) is free, the industry is characterized by constant returns to scale. The industry’s supply curve is horizontal so that \( AC = MC \) for all levels of output. In accordance with our assumptions, the competitive industry equilibrium is characterized by the following conditions:

\[ Q^* = f(X_1, \ldots, X_n) = \theta(\bar{p}^*) \]  
\[ \bar{p}^* = \bar{w}, \quad i = 1, 2, \ldots, n, \]  

where \( Q \) is the aggregate production function of the industry and \( \bar{p}^* \) is the negatively sloped relationship between quantity demanded and the certainty-equivalent output price. The conditions \( \bar{p}^* = \bar{w} \) are satisfied if the industry is perfectly competitive and therefore, the output elasticity of all inputs are equal.

The solution of the \( n \) + 1 equations (30) and (31) for \( \delta X_i / \delta b_0 \) yields:

\[ \frac{\partial \bar{q}}{\partial \bar{b}_0} = \lambda(X_i \bar{X}/P^*Q^*)(\eta - \sigma_i), \quad i = 1, 2, \ldots, n, \]  

where \( \sigma_i \) is Allen’s partial elasticity of substitution of the pair of factors \( X_i \) and \( X_j \). Equation (32) shows, clearly, that \( \partial \bar{X}_i / \partial \bar{b}_0 \geq 0 \) as \( \eta \geq \sigma_i \), and that for the special case of \( i = j \) we have

\[ \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) = \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) \]

\[ \frac{\partial \bar{q}}{\partial \bar{b}_0} = \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i), \quad j = 1, 2, \ldots, n. \]  

Since \( \sigma_i \) is negative by the definition of the partial elasticity of substitution, it follows from (33) that the aggregate industry demand for an input increases with an increase in the uncertainty of the input price.

Simplifying, the solution of the system of equations (30) and (31) for \( \delta \bar{q} / \delta \bar{b}_0 \) yields:

\[ \frac{\partial \bar{q}}{\partial \bar{b}_0} = \frac{1}{\bar{p}^*} \frac{\partial \bar{p}^*}{\partial \bar{b}_0}, \quad j = 1, 2, \ldots, n, \]  

and

\[ f_i(\frac{\partial \bar{p}^*}{\partial \bar{b}_0}) = f_i(\frac{\partial \bar{p}^*}{\partial \bar{b}_0}) + f_i(\frac{\partial \bar{X}_i}{\partial \bar{b}_0}) + f_i(\frac{\partial \bar{X}_j}{\partial \bar{b}_0}) + \ldots \]

\[ + f_n(\frac{\partial \bar{X}_n}{\partial \bar{b}_0}) = 1 \bar{w}, \quad i = 1, 2, \ldots, n, \]  

where \( \eta = (-\delta \bar{q} / \delta \bar{b}_0)Q^*/\delta \bar{b}_0 > 0 \) and \( \delta \bar{h}_i / \delta \bar{b}_0 = \lambda \) where \( i = 1 \) and zero otherwise. The solution of the \( n + 1 \) equations (30) and (31) for \( \delta \bar{X}_i / \delta \bar{b}_0 \) yields:

\[ \frac{\partial \bar{X}_i}{\partial \bar{b}_0} = \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i), \quad i = 1, 2, \ldots, n, \]  

where \( \sigma_i \) is Allen’s partial elasticity of substitution of the pair of factors \( X_i \) and \( X_j \). Equation (32) shows, clearly, that \( \partial \bar{X}_i / \partial \bar{b}_0 \geq 0 \) as \( \eta \geq \sigma_i \), and that for the special case of \( i = j \) we have

\[ \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) = \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) \]

\[ \frac{\partial \bar{X}_i}{\partial \bar{b}_0} = \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i), \quad j = 1, 2, \ldots, n. \]  

Equations (34) and (35) state that under uncertainty and risk aversion the aggregate output of the industry increases and the certainty-equivalent output price falls with an increase in the uncertainty of an input price. The sequence of events which produces these industry effects is easily traced through the adjustments of the representative firm. Since long-run equilibrium requires that \( \bar{p} \) adjusts to the minimum \( AC \) of the representative firm, equation (24) shows that \( \delta \bar{C}_i / \delta \bar{b}_0 = \delta \bar{p}_i / \delta \bar{b}_0 \) is an immediate consequence of this shift in \( AC \). That is, the minimum \( AC \) point of the representative firm shifts downward with an increase in \( \eta \) and, therefore the horizontal certainty-equivalent supply curve of the industry moves downward. Given the negative slope of the certainty-equivalent demand curve, a shift downward in the certainty-equivalent supply curve of the industry must reduce \( \bar{p} \) and increase \( Q \) as shown in equations (34) and (35). This sequence of events clarifies equations (32) and (33) as well. As \( \eta \) rises the demand for input factors is affected in two ways. Firstly, the certainty-equivalent cost of production is now lower and, therefore, \( \eta < 0 \). \( Q^* \) increases which in turn causes increase in the demand for all factors. This effect is shown by the positive term \( \lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) \). Second, the factor \( X_i \) is now relatively less expensive than other factors and its price to substitute \( X_j \) for other factors in production. The demand for \( X_i \) thus increases is account of substitution, as shown by the positive term \( -\lambda(X_i \bar{X}_j / P^*Q^*)(\eta - \sigma_i) \), as well as account of the output effect. The effect of a change in \( \eta \) on the demand for \( X_i \) is positive if \( \sigma_i < 0 \) and \( \delta \bar{h}_i / \delta \bar{b}_0 < 0 \).

Next we turn to study the impacts of a change in the uncertainty of output price. For this purpose we write the market equilibrium condition for a competitive industry as

\[ \bar{p}^* = \bar{D}(\bar{Q}^*, \bar{w}, \bar{w}, \ldots, \bar{w}), \]  

\[ \bar{D}(\bar{Q}^*, \bar{w}, \bar{w}, \ldots, \bar{w}), \]  

where \( \bar{D} = \bar{D}(\bar{Q}^*, \bar{w}, \bar{w}, \ldots, \bar{w}) \) is the negatively sloped (inverse) certainty-equivalent demand function and \( \bar{D} \) is the horizontal certainty-equivalent supply function of the industry. Differentiation of (36) with respect to \( \bar{b}_0 = \bar{b}_1, \ldots, \bar{b}_0 \) yields

\[ \delta \bar{d} / \delta \bar{b}_0 = \delta \bar{D}(\delta \bar{D} / \delta \bar{b}_0) \delta \bar{D} / \delta \bar{b}_0 + \delta \bar{D} / \delta \bar{b}_0 \delta \bar{D} / \delta \bar{b}_0 \]

\[ \delta \bar{d} / \delta \bar{b}_0 = \delta \bar{D}(\delta \bar{D} / \delta \bar{b}_0) \delta \bar{D} / \delta \bar{b}_0 + \delta \bar{D} / \delta \bar{b}_0 \]  

This gives the linear homogeneous production function of the industry: \( Q \) inputs are total factors and their employment is positively related to output.
< 0, \frac{\partial \pi}{\partial q_0} = \lambda < 0 and \delta \pi / \delta q^* = 0

by virtue of the assumptions of similar forms and free entry (exit) into the industry. Therefore,

$$\phi^* = \frac{\partial \pi}{\partial q_0}$$

$$\delta \phi^*/\delta q^* = \lambda \delta \phi^*/\delta q^*$$

$$\delta \phi^*/\delta q^* < 0 \quad (37)$$

and

$$\delta \phi^*/\delta q^* = 0 \quad (38)$$

which shows that under uncertainty and risk aversion the aggregate output of the industry decreases and the certainty-equivalent output price remains intact with an increase in the uncertainty of output price. The explanation of this result is straightforward. It states that an increase in \( \delta \) shifts \( \delta \) to the left while \( \delta \) remains intact and, therefore, \( Q^* \) falls while \( P \) is unaffected. The fact that \( \delta \) is unaffected is easily verified from equation (21) where \( \delta \phi^*/\delta q^* = \delta \phi^*/\delta q^* = 0 \).

Now we examine the response of the industry in terms of demand for inputs with respect to an exogenous change in the uncertainty of output price. Under our assumptions of pure competition, the same technology is available to all firms, exact and potential, so the industry is represented by a linear homogeneous production function. That is, the aggregate production function \( Q = f(X_1, \ldots, X_n) \) is homogeneous of the first degree. Therefore, by Shephard's lemma for homogeneous production functions, the demand for each input can be written in a multiplicatively separable form of \( X_i = G_i(q^*) A_i (\xi_1, \ldots, \xi_n) \) where \( G_i \) is a function of output prices only.\(^3\) Hence, for any two input factors \( X_i \) and \( X_j \) the factor proportions depend on the certainty-equivalent factor prices only, i.e., \( X^*_i / X^*_j = G_i(q^*) A_i (\xi_1, \ldots, \xi_n) / G_j(q^*) A_j (\xi_1, \ldots, \xi_n) = \beta_i (\xi_1, \ldots, \xi_n) \).

Therefore, \( \delta \phi^*/\delta q^* = 0 \). Applying the quotient rule of differentiation and rearranging terms, one obtains

$$\delta \phi^*/\delta q^* = \left( \frac{\partial}{\partial q^*} \right) \left( \frac{\partial \phi^*}{\partial \xi_i} \right) \delta \xi_i / \delta q^* \quad (39)$$

Differeneiation of \( Q^* = f(X_1, \ldots, X_n) \) with respect to \( \phi^* \) and utilization of (37), yields

$$\frac{\partial Q^*}{\partial \phi^*} = \sum_{i=1}^n X_i \frac{\partial X_i}{\partial \phi^*} < 0 \quad (40)$$

By equation (39) we have sign \( (\delta X_i / \delta q^*) = \text{sign} (\delta X_i / \phi^*) \) for all \( i \) and \( j \). Since \( \delta X_i / \delta q^* > 0 \), it follows from (40) that \( \delta X_i / \delta q^* < 0 \) for \( i = 1, 2, \ldots, n \). Thus equations (39) and (40) jointly establish that under uncertainty and risk aversion the industry's demand for all input factors decreases proportionally with an increase in the uncertainty of output price. This result is in agreement with one's intuition. An increase in the uncertainty of output price reduces the output of the industry. Reduction in output, given the homothetic production function of the industry, calls for an all-round and proportional decrease in the employment of the factors.

A comparison of the results of sections IV and V shows a potential incongruence, in terms of signs (directions) and magnitudes, between the inputs and output adjustments of the representative firm and those of the industry. For example, it was shown, that increased uncertainty of an input price increases the industry demand for the input but the impact on the demand for this input by the representative firm could be positive or negative. Of course, any gap between the change in the quantity demanded by the industry and the sum of the changes in the quantity demanded by firms presently in the market must be picked-up by the entry (exit) of firms into from the industry. Given our assumption that all firms, exact and potential, are identical, the equilibrium output of the industry is the product of the equilibrium number of firms \( n^* \) times the equilibrium output of the representative firm, i.e., \( Q^* = n^* q^* \). But the derivative of \( n^* \) with respect to \( \phi \) is

$$\frac{dn^*}{d\phi} = \frac{1}{q^*} \left( \sum_{i=1}^n \frac{\partial X_i}{\partial \phi^*} q^* \right) \left( 1 - e_{\phi} \right)$$

Now we define the output elasticity of the certainty-equivalent marginal cost of the representative firm as \( e_{\phi} = s^* q^* M^* / M^* \). Then substituting the equilibrium relations \( P = M^* = n^* q^* \) and \( X^* = n^* q^* \) into equation (42) and rearranging terms, we obtain\(^4\)

$$\frac{dn^*}{d\phi} = \left( \frac{\partial X_i / \partial q^*}{\partial q^*} \right) \left( 1 - e_{\phi} \right) \quad \text{for} \quad j = 1, 2, \ldots, n \quad (43)$$

Equations (43) show, instantly, that under uncertainty and risk aversion \( d\pi^*/d\phi \geq 0 \) for \( j = 1, 2, \ldots, n \) and \( e_{\phi} \geq 1 \). The sequence of events which leads to the entry (exit) of firms into the industry is easily traced through the respective output adjustment of the representative firm and the industry with respect to changes in the uncertainty of an input price. First, note that, since \( e_{\phi} \) is positive by the value maximization (cost-minimization) hypothesis, entry of firms occurs whenever \( e_{\phi} \leq 1 \). This entry of firms with respect to an increase in \( \phi \) is explained by the fact that \( d\pi^*/d\phi \geq 0 \), by equation (35), while \( d\pi^*/d\phi \geq 0 \) even if \( e_{\phi} \geq 1 \) as shown in equation (27). The industry is expanding output while the existing firms are contracting their output. Therefore, entry of new firms must occur in order to supply the industry's expansion as well as the contraction of output by firms presently in the industry. Second, equation (43) shows that \( d\pi^*/d\phi > 0 \) for any \( e_{\phi} > 1 \). As \( e_{\phi} \) approaches infinity, even when \( e_{\phi} > 1 \). As \( e_{\phi} \) approaches infinity, even when

\(^3\)Shephard (1953), section 7.

\(^4\)Sundaresan (1971) envisages a long-run equilibrium where firms owned by the more risk averse entrepreneurs will be marginal firms in the sense that they will be the first to leave the industry under adverse changes in market conditions. Our equations (41) and (43) show that exit from the industry is independent of owner's risk preferences.
of the representative firm becomes vertical and its output remains intact with respect to changes in \( \theta \), as shown by equation (27). Since \( 2\theta^* / \theta > 0 \) and \( \theta^*/\theta = 0 \) when the MC curve is vertical, entry of firms must take place in order to supply the expansion in the output of the industry. Finally, when \( \epsilon_{e,2} > 1 \) and \( \epsilon_{c,2} > 0 \), the expression \(-1 + \epsilon_{e,2} / \epsilon_{c,2} \) is positive and the sign of \( \partial y / \partial \theta \) depends on the numerical magnitude of this expression relatively to the numerical value of \( \theta \). The dependence of the sign of \( \partial y / \partial \theta \) on these relative magnitudes is explained by the fact that equation (27) shows that the output of the representative firm increases with an increase in \( \theta \) if \( \epsilon_{e,2} > 1 \), and that the rate of increase in output depends on the numerical value of \(-1 + \epsilon_{e,2} / \epsilon_{c,2} \) in a measure of the elasticity of supply of output of the representative firm while \( \eta \) represents the elasticity of demand for output by the industry. Since both, the industry (equation (35)) and the representative firm (equation (27) for \( \epsilon_{e,2} > 1 \), expand output with an increase in \( \theta \), equation (43) shows that the number of firms in the industry decreases (increases) if the supply of output by the firms presently in the market is sufficiently elastic (inelastic) to catch up with the output expansion of the industry which depends on \( \eta \).

VI. Conclusions

Our value maximizing firm and the "entrepreneurial" models represent two distinct approaches to the theory of the firm under uncertainty. The former approach assumes that the owners of a firm hold diversified portfolios rather than a position in a single firm. Additional assumptions include the existence of perfect and frictionless financial markets where either returns on securities are normally distributed or investors are characterized by quadratic utility functions. The latter approach allows more general owner preferences and distributions of returns but assumes that the firm is held by a single owner whose entire wealth is tied to its operations. The above approaches to the theory of the firm assume different market settings. While both approaches view the firm as a participant in the product and factor markets, the main difference is in the absence of capital markets within the "entrepreneurial" framework. That is, our value maximizing firm faces a given capital-market price of risk through which certainty-equivalent (market) values are computed. Consequently, the firm chooses its production plan with respect to a set of certainty-equivalent prices of output and inputs analogously to the case of complete markets. However, in the absence of a market mechanism for the valuation of risky claims, the "entrepreneurial" firm values uncertain prospects through the owners' subjective marginal rate of substitution between expected return and risk as in the case of incomplete markets. This difference between the objective (market) and subjective (utility) valuation of risky claims accounts for the deviation of the results of the "entrepreneurial" firm from our results and from the standard certainty results. The "entrepreneurial" firm faces two distinct effects as a consequence of a shift in any of the market parameters which are embedded in its random profit function. The first effect is objective and it is reflected in a shift of its transformation curve between expected profits and risk. The second effect involves a change in the subjective "price of risk." That is, as the transformation curve shifts, a new tangency point between an indifference curve and the expected profit-risk transformation curve is obtained. Since the marginal rate of substitution (the subjective "price of risk") at the new tangency point is generally different from the one which corresponds to the initial tangency, it follows that the subjective "price of risk" varies in response to changes in market parameters. Therefore, the comparative static results of the "entrepreneurial" firm reflect the joint impact of a change in a particular market parameter combined with a change in the subjective "price of risk." Our value maximizing firm faces a constant market price of risk and, therefore, it is subject only to the shift in the expected profit-risk transformation curve. Thus the comparative static results of our firm are the pure consequences of shifts in market parameters only. Furthermore, within our framework the value maximizing production plan is optimal in the sense that it is unanimously preferred by all shareholders. The main conclusions concerning the value-maximizing firm under uncertainty are summarized in the next exhibit and in the following two paragraphs.

Exhibit 1: Equilibrium Responses to Changes in Uncertainty

<table>
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<tr>
<th>Short Run (Section III)</th>
<th>Long Run (Section IV)</th>
<th>Industry (Section V)</th>
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<td>( \delta^*/\delta )</td>
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<td>( \delta^*/\delta )</td>
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<td>( \epsilon_{e,2} \geq 1 )</td>
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<td>( \epsilon_{e,2} &lt; 0 )</td>
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Sections III and IV examine the impacts of uncertainty and risk aversion on the behavior of the firm in short-run and long-run equilibrium settings, respectively. It is shown that, in a short-run equilibrium, an increase in the uncertainty of an input price produces the following effects: (1) increased demand for the input; (2) increase in the market value of the firm; and (3) increase (decrease) in the optimal output of the firm if the input is a normal (inferior) factor. The consequences of an increase in the uncertainty of output price are: (1) decrease (increase) in demand for normal (inferior) inputs; (2) decrease in output; and (3) decrease in the market value of the firm. Section IV shows that a competitive firm in a long-run equilibrium earns zero certainty-equivalent profits, but its expected profits are positive or negative depending on whether the uncertainty of output price is greater or smaller than the uncertainty of average cost. Furthermore, in this setting, an increase in the uncertainty of an input price generates the following results: (1) the average product of the input declines; (2) the demand for the input increases if the output elasticity of this input is greater than unity; and (3) the optimal output of the firm increases (decreases) if the output elasticity is greater (less) than unity.

Section V examines the equilibrium adjustments of the industry under uncertainty and risk aversion. It is shown that an increase in the uncertainty of output price reduces the number of firms in the industry and causes a contraction in aggregate output which is accompanied by a decline in the industry's demand for every input factor. However, increased uncertainty of an input price induces...
expansion in the employment of this input as well as an increase in the output of the industry. Finally, conditions, in terms of elasticities of demand and supply, are established for the effect of input price uncertainty on the exit (exit) of firms from (from) the industry.

An additional advantage of the value maximization approach to the theory of the firm under uncertainty is that it provides a complete markets setting where the problem of aggregation of inputs and output, across firms into industry schedules, is easily solved (section V). Moreover, many of the comparative static results are unambiguous and empirically testable. Finally, our capital market oriented theory of the firm links the real and financial sectors of the economy. This linkage provides a potentially important channel through which macroeconomic policies affect an economy which is operating under conditions of uncertainty and risk aversion. Thus it may be possible to trace the impacts of monetary policy, via the capital market, on output and on the employment of labor and capital. Likewise, the impacts of fiscal policies on the risk-return combinations which are priced in the financial market may be tractable through this linkage.

References