I. Introduction

Under the putty-clay hypothesis, it is of interest to identify and estimate the ex ante production possibilities available to the firm from observed ex post data. Fuss [1] and Fuss and McFadden [2] have shown that the ex ante production technology can be identified through the choice of technique (or design) of the firm. In Fuss [1], it is assumed that there is no uncertainty in prices. In Fuss and McFadden [2], the choice of technique is considered under uncertainty in prices. However, their framework assumes that the expected profit function of each technique is linear in the technique-specific parameters.

In this paper, we re-examine the choice of techniques for a risk-neutral firm under uncertainty in prices. Under the assumption of expected cost minimization, the optimal technique can be derived by certainty equivalence at the means of the prices. However, under the assumption of expected profit maximization and gamma distributed prices, the optimal technique is the same as that of certainty equivalence at the means if and only if the ex ante production function is of a specific Cobb-Douglas form. It is also shown that the expected profit function of each technique is not linear in the technique-specific parameters.

II. The Model

We assume that, other than capital, there are n variable inputs denoted by x₁, ..., xₙ. Capital is heterogeneous. The production technology embodied in each type of capital equipment is characterized by a fixed maximum output capacity, and fixed input coefficients which are the quantities of variable inputs required per unit output. A given vector of input coefficients, denoted by a = (a₁, ..., aₙ), is referred to as a technique. We assume constant...

*Stanford University, Palo Alto, California, and McGill University, Montreal, Quebec, respectively. The authors wish to thank Professor John Chipman, Melvyn Fuss, Luen-Fai Lee and Daniel McFadden for their comments. The research of the second author was supported in part by a grant from the Social Sciences and Humanities Research Council of Canada.

1For a discussion of the putty-clay hypothesis, the reader is referred to Fuss [1] or Fuss and McFadden [2].

Fuss [1] and Fuss and McFadden [2] in fact show that the ex ante production technology can be identified in a more general putty-semi-putty model.

We assume that it is feasible to produce an amount of output up to the maximum output capacity.
capital cost per unit output capacity, i.e., capital cost per unit output capacity does not vary with output capacity nor across techniques. 4

Under our assumptions, the ex ante production function exhibits constant returns and may be written as:

\[ y = F(x_1, \ldots, x_a), \]

where \( y \) is the quantity of output and \( F(x_1, \ldots, x_a) \) is linear homogeneous. 5

A given technique \( a \) is simply a point on the units isoquant of the ex ante production function. It is more convenient to represent a in the "price" space by a change of variables: let \( G(p, y) \equiv k(p, y \bar{a}) \) be the ex ante cost function dual to the ex ante linear homogeneous production function \( F(x_1, \ldots, x_a) \), where \( y = (p_1, \ldots, p_a) \) is the price vector. \( k(p) \) is then the ex ante unit cost function. Assuming \( k(p) \) is twice continuously differentiable and that its Hessian matrix has rank \( n-1 \), then for a given technique \( a \), there exists a price vector \( q \), unique up to a scalar multiple, 6 such that

\[ \frac{dk(q)}{dp} = k(q) = a_i, \quad i = 1, 2, \ldots, n. \]  

(1)

The vector \( q \) may be referred to as the "planning" price vector of the technique \( a \). It is clear that any given technique \( a \) can be equivalently represented by the corresponding planning price vector \( q \).

III. Expected Cost Minimization

The input prices are assumed to be nonnegative random variables with known distributions. The firm's problem is to find the type of capital equipment with the minimum expected cost of producing a given quantity of output. For a given technique \( a \) and input price vector \( p \), the unit cost of production is:

\[ \sum_{i=1}^{n} p_i q_i. \]

The expected cost of production may be obtained as

\[ \sum_{i=1}^{n} p_i a_i. \]

where \( p_i = E(p_i), i = 1, 2, \ldots, n \). Hence the firm's problem becomes

\[ \min_{a} \sum_{i=1}^{n} p_i q_i \text{ s.t. } F(a) = 1 \]  

(2)

which is simply the familiar cost minimization problem with input prices equal to their means with certainty. It follows that the optimal technique can be derived by certainty equivalence at the means.

IV. Expected Profit Maximization

Prices are assumed to be independent, 7 nonnegative random variables with known distributions. The output price has an exponential distribution and input prices have gamma distributions. 8 The firm has a fixed capital budget and its problem is to find the type of capital equipment (subject to the capital budget) with the maximum expected profit. We seek whether the optimal technique can also be derived by certainty equivalence at the means.

For a given output price \( p_0 \), input price vector \( p \), and technique \( a \), profit is given by:

\[ \prod_{i=1}^{n} (p_i, p; a) = \max \left\{ 0, p_0 - \sum_{i=1}^{n} a_i q_i \right\}. \]  

(3)

The form of (3) reflects the fact that a profit-maximizing firm will simply not operate when profit is negative. The expected profit may be obtained by the following lemma:

**Lemma.** Let the distribution of \( p_0 \) be exponential with density function \( a_0 \exp(-a_0 p_0) \), \( a_0 > 0 \), and the distribution of \( p_i \) \( (i = 1, 2, \ldots, n) \) be gamma with density function

\[ a_i^\alpha_i \exp(-a_0 p_i)/\Gamma(n_i), \]

where \( \Gamma() \) is the gamma function, \( a_i > 0 \), and \( n_i > 0 \); then the expected profit is given by

\[ E\left( \prod_{i=1}^{n} (p_i, p; a) \right) = \mu_0 / \prod_{i=1}^{n} (1 + \mu_i a_i/\mu_i n_i)^{n_i} \]

where \( \mu_i = a_i / a_0 = E(p_i), i = 1, 2, \ldots, n \) and \( \mu_0 = E(p_0) = 1/a_0 \).

In the terminology of Fuss and McFadden [2], the \( a_i \) \( (i = 1, 2, \ldots, n) \) in the expected profit function \( E(\prod_{i=1}^{n} (p_i, p; a)) \) are the technique (design)-specific parameters. It is apparent that \( E(\prod_{i=1}^{n} (p_i, p; a)) \) is not linear in the \( a_i \).

The firm's problem is to find the technique with the maximum expected profit:

\[ \max_{a} \mu_0 / \prod_{i=1}^{n} (1 + \mu_i a_i/\mu_i n_i)^{n_i} \]

s.t. \( F(a) = 1 \)  

(4)

Using the fact that \( a \) can be represented by the corresponding planning price vector \( q \), (4) may be rewritten as:

\[ \max_{a} \sum_{i=1}^{n} \mu_i q_i / \prod_{i=1}^{n} (1 + \mu_i a_i/\mu_i n_i)^{n_i} \]

over the region \( q > 0 \). If for any given \( \mu > 0 \), \( q = \mu (p_1, \ldots, p_n) \) solves (5), the optimal technique can be derived by certainty equivalence at the means. We now state the principal result of our paper:

**Theorem.** Suppose the Heaviside matrix of the unit cost function,

\[ \delta \theta^2 (\delta y/\delta p; \bar{y} \bar{a}) = H(q) \quad i, j = 1, 2, \ldots, n \]

(6)

The assumption that prices are statistically independent of one another does not exclude the possibility that the means of the prices may be related to one another. The exponential distribution is a special case of gamma distribution. Hence all prices are gamma distributed.

(For a proof of the Lemma, see Ma [4].)
has rank \( n - 1 \), then for any given \( \mu > 0 \), \( q = \mu \) solves (8) if and only if

\[
A \prod_{i=1}^{n} q_i^{a_i / 0} = \prod_{j=1}^{m} h_j(q) / n \mu h_j(q)
\]  

where \( A \) is an arbitrary positive constant.

Proof: It is sufficient to show that for any given \( \mu > 0 \), the minimum of

\[
\prod_{i=1}^{n} (1 + \mu_i h_i (q)) / n \mu h_i (q)
\]

is attained at \( q = \mu \) if and only if \( h_i (q) \) is of the form in (6).

The first order conditions for a minimum of (7) are:

\[
\sum_{j=1}^{n} (\mu_i / n \mu_i h_j (q)) \cdot \frac{d h_j (q)}{d q_j} = 0, \quad j = 1, 2, \ldots, n
\]

If \( q = \mu \) attains the minimum, \( q = \mu \) or any scalar multiple of \( \mu \) solves (8). Let \( q = \mu / \mu_0 \), then (8) becomes:

\[
\sum_{j=1}^{n} \left( \frac{q_j}{1 + (q_j / n \mu_0 h_j (q))} \right) \cdot \frac{d h_j (q)}{d q_j} = 0, \quad j = 1, 2, \ldots, n.
\]

Linear homogeneity of \( h() \) implies that:

\[
\sum_{i=1}^{m} q_i / \lambda = 0, \quad j = 1, 2, \ldots, n.
\]

But the Hessian matrix is of rank \( n - 1 \), hence (9) and (10) together imply that:

\[
q_i / \left( 1 + (q_i / n \mu_0 h_i (q)) \right) = \lambda q_i, \quad i = 1, 2, \ldots, n.
\]

where \( \lambda \) is a scalar, which simplifies to

\[
\lambda (q_i / n \mu_0 h_i (q)) = n_i, \quad i = 1, 2, \ldots, n.
\]

Summing (12) over \( i \), and using the fact that \( h() \) is linear homogeneous, we obtain:

\[
\lambda = \left( \sum_{i=1}^{n} n_i \right) / \left( h(q) + \sum_{i=1}^{n} n_i \right).
\]

Substituting (13) into (12) and simplifying, we obtain:

\[
q_i h_i (q) = n_i h_i (q) / \sum_{j=1}^{n} n_j, \quad i = 1, 2, \ldots, n
\]

which may be integrated to yield (6).

To prove the converse, we show that if \( h(q) \) is of the form given in (6), then \( q = \mu \) minimizes (7), i.e.,

\[
\prod_{i=1}^{n} (1 + (\mu_i / \sum_{j=1}^{m} q_j a_j / \sum_{j=1}^{m} a_j n_j))^{a_i} \\
\geq \prod_{i=1}^{n} (1 + (1 / \sum_{j=1}^{m} n_j) a_j / \sum_{j=1}^{m} a_j n_j)^{a_i}
\]

for any \( \mu > 0 \). Taking the \( (\sum_{j=1}^{m} n_j) \)th root of both sides and simplifying, we obtain:

\[
\prod_{i=1}^{n} (1 + (\mu_i / \sum_{j=1}^{m} n_j) a_j / \sum_{j=1}^{m} a_j n_j)^{a_i} / \sum_{j=1}^{m} n_j
\]

By an inequality theorem in Hardy, Littlewood, and Polya [6] (p. 21, theorem 10), we have, if \( a_i, b_i, c_i (i = 1, 2, \ldots, n) \) are nonnegative numbers and

\[
\sum_{i=1}^{n} c_i = 1
\]

then the following inequality holds:

\[
\prod_{i=1}^{n} (a_i + b_i)^{c_i} \geq \prod_{i=1}^{n} a_i^{c_i} + \prod_{i=1}^{n} b_i^{c_i}
\]

(16)

Let \( a_i = 1, b_i = (\mu_i / \sum_{j=1}^{m} n_j) a_j / \sum_{j=1}^{m} a_j n_j \) and \( c_i = n_i / \sum_{j=1}^{m} n_j, i = 1, 2, \ldots, n \), then (16) can be seen to be implied by (16). Q.E.D.

The theorem says that for independent gamma distributed prices there is only one ex-ante unit cost function of the Cobb-Douglas form (up to an arbitrary multiplicative constant \( \lambda \)) and hence by duality is ex-ante production function of the Cobb-Douglas form for which the optimal technique can be derived by certainty equivalence at the means under the assumption of expected profit maximization.

V. Concluding Remarks

We have shown that the optimal technique under expected profit maximization is the same as that described in this paper if there exists a certainty equivalent for the gamma distributed prices. However, it is possible to extend this result to the case where the prices are gamma distributed, in which case the optimal technique is given by the certainty equivalent at the means under the assumption of expected profit maximization.

Empirically, the result implies that the specification of the optimal choice of technique is that which is most consistent with expected profit maximization. A second approach, proposed by Fuss and McFadden [2], appears to require the assumption that the expected profit function is linear in the technique-specific parameters, which cannot be expected to hold in general. Perhaps a third approach, based on maximizing an explicit closed form expression for the expected profit function, such as one derived in this paper, subject to a production function, may be useful in generating an estimable econometric specification.

References

Charges, Permits and Pollutant Interactions

Alfred Endres*

I. Introduction

There has been considerable discussion in the literature as to how predetermined environmental quality standards can be attained. Efficient charges and transferable discharge permits (TDPs) are the policy instruments considered most often. In most of the literature the problem is treated as if different pollutants could be regulated independently. However, even though a useful assumption to facilitate theoretical analysis in the first place, this approach ignores many important aspects of practical pollution problems.

Generally, the environment does not provide special subcapacities for the assimilation of each pollutant. Several pollutants rather draw upon the same capacity of the environment, simultaneously. Moreover, they often react chemically. These mixtures then generate an environmental impact different from the sum of the impacts that individual pollutant would have in the absence of the others.

In all three cases, an acceptable level of a pollutant can only be defined for given levels of other pollutants.

In the following analysis it is assumed that a single indicator “I” exists which relates the quantities of n pollutants \(X_1, \ldots, X_n\) to “load units” of this medium. (The higher the index value the lower is the quality of the environmental medium). This indicator is assumed to take care of the problems of simultaneous environmental capacity use and chemical reactions.

Here, the target of environmental policy can be defined in terms of a predetermined level \(I\) of this index. It should be noted that this type of a target definition, contrary to the pollutant specific definition traditionally used, is compatible with indefinitely many combinations of \(n\) pollutant quantities.

Of course, it cannot be said in general terms what properties the environmental constraint defined for the economic process by setting the target \(I\) might have. If the indicator would take the linear additive form of \(I = a_1X_1 + \cdots + a_nX_n\), where the \(a_i\) are constant “load-parameters”, the pollutants could be substituted against each other at a constant rate for each given level \(I\). This very simple type is called “linear interaction”, below. Of course,

*Technical University of Berlin, West Berlin, West Germany.

1 Notable exceptions are [2] and [3].