Optimal Population Growth and The Social Welfare Function

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INTRODUCTION

The modern theory of optimal economic growth has focused almost total attention on two types of utilitarian models, those that maximize per capita utility, and those that maximize total utility. Starting with Ramsey (1928), the optimality criterion most frequently used has been the per capita lifetime utility criterion.

In the area of optimum population, Mill based his justification for limiting the size of population on the principle of maximizing per capita utility. Samuelson (1975) followed the same approach as he sought to determine the optimal growth rate of population in a two-period overlapping generations model. The total welfare criterion also known as the classical utilitarian criterion, which weights individual utilities by the number of people in society, is characteristic of Bentham, Paley, and Sidgwick, and has been worked into formal economic models by Maae (1955), Dasgupta (1969) and Lane (1977).

A welfare criterion intermediate between the above two formulations, is adopted in this paper to determine its effects on optimal population policy and the optimal capital accumulation rule. It is the social welfare function \( V(c,L) \) which includes the size of population \( L \) as an argument of that function, in addition to per capita consumption. We are postulating that societal welfare depends not only on the level of per capita consumption, but also on the population density applicable in the area in which the individual lives. The latter may be taken as a proxy for quality of life characteristics, such as the quality of education, environmental factors such as clean air, pure water, outdoor recreational facilities, and less tangible attributes such as individual-political and consumer sovereignty (Spengler, 1966, p. 13). Our social welfare function \( V(c,L) \) is increasing in \( c \), initially increasing in \( L \) and then declining in it due to crowding.

Even though the need for the adoption of such a welfare criterion has been acknowledged in the literature, its theoretical implications for optimal growth theory have not as yet been fully investigated in the published literature.1 Our paper attempts to bridge this gap.

Utilizing the \( V(c,L) \) optimality criterion, the maximization of the discounted social welfare function of all generations with respect to both \( c \) and \( L \) is carried out under the assumption of a constant returns to scale technology (Case 1) and a decreasing returns to scale technology (Case 2). We derive our own version of the Meade Rule for optimal population growth; we also find that an optimal growth rate \( \gamma^* \) exists in our model and allows for capital accumulation along the modified Golden Rule path. The results suggest that our maximizing \( V(c,L) \) model bridges the per capita and total utility approaches to the problem of optimum population.2 It is relevant that these results have not been obtained by Nerlove and his coauthors (1982, 1985), who have even the past several years or so worked in different ways with this welfare criterion.

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CASE I: CONSTANT RETURNS TO SCALE TECHNOLOGY

The Model

The objective of this model is to find the optimal per capita consumption (c) and population size (L) profiles that maximize the discounted social welfare for all time under the assumption that population is completely controllable and without cost. Production takes place under constant returns to scale. It is assumed that the function \( f(t) \) is increasing, strictly concave, twice differentiable and that it satisfies the Iwasawa derivative conditions. The objective function is assumed to be increasing and strictly concave in \( \psi \) with

\[
\lim_{\psi \to -\infty} \psi' = -\infty, \quad \lim_{\psi \to -\infty} \psi = -\infty
\]

It is initially increasing and then declining in \( L \), i.e.,

\[
u_1 > 0 \text{ for } L < \bar{L}, \quad \nu_1 = 0 \quad \nu_1 < 0 \text{ for } L > \bar{L}
\]

with

\[
u_2 < 0 \text{ and } \lim_{\psi \to \infty} \psi_2 = -\infty, \quad \lim_{\psi \to \infty} \psi = -\infty.
\]

Thus, for a given level of per capita consumption \( \psi \), total utility increases with increasing density, reaches a maximum and then declines with further increases in population due to the diseconomies of "overcrowding." This is shown diagrammatically in Figure 1.\(^5\) The function chosen for our analysis is the following:  

\[
u(\psi, L) = a \log \psi + b \log L - \frac{b}{2} L^2
\]

with \( \psi_1 = \frac{a}{b}, \quad \psi_2 = b \left( \frac{1}{2} + \frac{1}{L} \right) \).

This is a semi-additively separable function in \( \psi \) and \( L \) which satisfies all of the above properties, including strict concavity. The parameters \( a \) and \( b/a \) measure the distance between the iso-utility curves of \( \psi \) versus \( L \) and the consumption-crowding trade-off, respectively. Function (2.1) is robust for a fairly wide range of the values of the parameters \( a \) and \( b \). Figure 2 depicts the utility indifference curves for this function.

\[
\text{Figure 1}
\]

In this model, the discounted social welfare function is maximized with respect to \( \psi \) and \( L \):

\[
(2.2) \quad \max_{\psi, L} \int_{t_0}^{t} e^{-\rho(t)} u(\psi(t), L(t)) \, dt \quad \rho > 0
\]

subject to the following capital accumulation constraint:

\[
(2.3) \quad \dot{K}(t) = f(K(t)) - c(t)
\]

with \( c(0) = 0, K(0) = 0, L(0) = 0 \) and with the initial capital stock specified:

\[
(2.4) \quad K(0) = K_0.
\]

The rate of growth of population \( s = \frac{d}{L} \); the state equation of capital per capita is given by:

\[
(2.5) \quad \dot{K}(t) = f(K(t)) - (c(t) - n(t)K(t)).
\]

This is a one-state \( (K) \) variable problem. Population is instantaneously adjusted to its optimum level relative to the capital stock, as in the Dasgupta (1969) problem. We express the Hamiltonian both in the current value form \( (H) \) and in the discounted value form \( (\hat{H}) \) and optimize the latter using Pontryagin's (1962) maximum principle.

\[
(2.6) \quad H = u(c, L) + w_k (L/k - c)
\]

\[
(2.7) \quad \hat{H} = e^{-\rho t} u(c, L) + \rho_k (L/k - c)
\]

where \( \rho_k = e^{-\rho t} \rho_c \) and \( \rho_k \) is the marginal (shadow) price of capital. The first order (necessary) conditions for an interior optimum are:

\[
(2.8) \quad e^{-\rho t} \frac{\partial u}{\partial \psi} - 1 = \rho_k = \rho_c
\]

and

\[
(2.9) \quad e^{-\rho t} \frac{\partial u}{\partial L} = e^{-\rho t} w_k (c - [f(k) - f(L)]).
\]

Equation (2.9) describes our version of the Meade Rule for optimal population growth, which we call the modified Meade Rule. According to this rule, the size of population is optimal, when the increase in the (discounted) social welfare due to the introduction of a new
member (u_t, L) just equals the reduction in (discounted) welfare brought about by that addition to population (u_t(c) - [k_t(k) - k(t)]). This cost is given as in the original Meade Rule by the difference of the new member's consumption and his marginal product, this difference multiplied by the marginal utility of consumption. At the optimum, the shadow price of labor is equal to zero. This is implied by the necessary condition (Lame, 1975, p. 59).

The optimal capital accumulation rate for this model is the modified Golden Rule. We obtain the equation for the percentage change in (discounted) marginal utility by differentiating both sides of the first order condition of eq. (2.8): (See Appendix for derivation)

\[ f'_s = e^{(n-\rho)} + u'_t \cdot e^{(n-\rho)} = n - f'(k) \]

Therefore

\[ \frac{f'_s}{f_s} = f'(k) - n. \]

Thus, the percentage rate of fall in discounted marginal utility is equal to the marginal product of capital minus the rate of population growth.

Solving eq. (2.10) for (e), we have

\[ c = \frac{u_t}{u'_t} [n + \rho - f'(k)] \]

Condition (2.11) (or, equivalently, (2.12)) states that capital is accumulated until the rate of interest (f'(k)) is equal to the sum of the growth rate of population and the social rate of time preference (n + \rho). This is the modified Golden Rule, which also applies where per capita utility is maximized.

**Steady State**

At steady state c = k = 0. Setting k = 0 and solving (2.5) for the optimal (n) we have:

\[ n^* = \frac{f'(k^*) - c^*}{k^*} \]

Setting c = 0,

\[ f'(k^*) = n + \rho. \]

Thus, the optimal steady state capital-labor ratio (k^*) is the solution to f'(k^*) = n + \rho and changes as (n^*) changes. In Dasgupta's (1969) total utility model, optimal capital accumulation is determined according to the Ramsey Rule (f'(k^*) = \rho) and is, therefore, independent of (n^*).

Combining the modified Meade Rule of eq. (2.9) as steady state and eq. (2.13) we solve for (n^*). The maximization of the discounted value of u(c, L) implies u_t > 0 and, therefore, c > |f'(k^*) - k(t)|. Substituting for (c^*) in eq. (2.13), we find n^* < f'(k^*). Thus, an optimal growth rate of population (n^*) exists in our model and is less than the equilibrium marginal product of capital. In Dasgupta's (1969) model, the optimal growth rate (n^*) is found to be less than the social rate of time preference (\rho). When he incorporates negative population externalities in his model, but still maintains the assumption of constant returns to scale, he finds that the system tends toward a zero growth optimal steady state, if the population externalities are "large enough" (Dasgupta, 1969, p. 313).

We consider the case of a^* = 0, i.e., the optimal stationary steady state, if it existed. A stationary state implies (from eq. (2.13)) that f'(k^*) = c^*, i.e., zero savings. Substituting for (c^*) in (2.9) and solving for (L^*), we have

\[ L^* = \frac{\alpha}{\lambda} [k^* f'(k^*)] \]

Let the production function be a Cobb-Douglas function homogeneous of degree one. In per capita terms:

\[ f(k) = k^n \]

with

\[ f'(k) = nk^{n-1} \]

Substituting in eq. (2.15) for f'(k^*) from (2.16) and for (u_t^*) and (u_t^*) from (2.1), we have

\[ L^* = \frac{\alpha}{\lambda} [\frac{\alpha}{\lambda}]^{1/2} \]

Similarly, we obtain a stationary steady value for (e):

\[ c^* = f'(k^*) - k^{n-1} \]

Now, from equations (2.14) and (2.16), when n = 0

\[ k^* = \frac{\alpha}{\lambda} \]

Therefore,

\[ c^* = f'(k^*) - \frac{\alpha}{\lambda} \]

Equation (2.18) shows that zero growth steady state per capita consumption and the discount rate are inversely related. We solve for (K^*) by setting (K) in eq. (2.3) equal to zero and substituting for (c^*) from eq. (2.18) and for (L^*) from eq. (2.17):

\[ K^* = \frac{\alpha}{\lambda} [\frac{\alpha}{\lambda}]^{1/2} \]

Assuming the following values of the parameters:

\[ a = 1.7 \quad b = 1.2 \quad \alpha = 0.25 \quad \rho = 0.3 \]

we obtain the following numerical values:

\[ L^* = 0.804 \quad c^* = 2.02 \quad K^* = 1.358 \]

The sufficiency proof and stability about the optimum for this model have been carried out.

**CASE 2: DECREASING RETURNS TO SCALE TECHNOLOGY**

Case 2 seeks to determine the optimum level of population when production takes place under decreasing returns to scale that are attributable to land and natural resource constraints.
In this case, the system tends to a steady state with zero growth and a fixed optimal value for population in the long-run.

The decreasing returns to scale technology is represented by a Cobb-Douglas production function homogeneous of degree less than unity, i.e.,

\[ F(K, L) = K^{\alpha}L^{\beta} \]

with \( \alpha, \beta > 0 \) and \( \alpha + \beta < 1 \). Equation (2.3) becomes

\( K = K^{*}L^{\frac{\beta}{1-\beta}} - cL \)

The necessary conditions for an interior optimum are analogous to those obtained under constant returns to scale:

\[ e^{-\alpha u'_c} - L \frac{\partial u_c}{\partial L} = \frac{\partial u_c}{\partial K} \]

and

\[ e^{-\alpha u'_c}L - e^{-\alpha u_c}[c - \alpha K^{*}L^{\alpha - 1} \beta] \]

The percentage rate of decline of (discounted) marginal utility per capita is given by equations (3.5) and (3.6), below:

\[ -\frac{\partial u_c}{\partial K} - \alpha K^{*}L^{\frac{\beta}{1-\beta}} - \beta \]

and

\[ -\frac{\partial u_c}{\partial K} - \alpha K^{*}L^{\alpha - 1} \beta - \rho \]

Equation (3.6) indicates that optimal capital accumulation follows the Ramsey Rule. Since \( K^{*} \) is zero, the marginal product of capital \( (\alpha K^{*}L^{\alpha - 1}) \) is equal to the positive rate of time preference \( \mu \).

**Steady State**

Steady state zero growth is determined by setting \( \dot{K} = 0 \), which yields

\[ K^{*} = e^{\frac{\alpha}{1-\beta}L^{\frac{\beta}{1-\beta}}} \]

and

\[ c^{*} = K^{*}L^{\alpha - 1} \]

Thus, steady state per capita consumption is equal to per capita output, i.e., savings is zero and the level of population is constant. We, therefore, solve for the optimal stationary state values of \( L \), \( c \), and \( K \). Substituting in (3.4) for \( u'_c \) and \( u'_c \) from (2.1) and for \( c \) from (3.8), we have:

\[ L^{*} = \left[ 1 - \frac{\beta}{\alpha} (1 - \beta) \right]^{\frac{1}{1-\beta}}. \]

Substituting in (3.7) for \( L \) from (3.9) and solving, we have:

\[ K^{*} = \frac{\beta}{\alpha} \left[ 1 - \frac{\beta}{\alpha} (1 - \beta) \right]^{\frac{1}{1-\beta}}. \]

Finally, we obtain an expression for \( c^{*} \) in terms of the parameters of the system. We substitute in (3.8) for \( K \) from (3.10) and for \( L \) from (3.9). Therefore,

\[ c^{*} = \frac{e^{\frac{\alpha}{1-\beta}L^{\frac{\beta}{1-\beta}}} - \frac{\beta}{\alpha} (1 - \beta)}{\frac{\beta}{\alpha} (1 - \beta)} \]

Assuming the same values of the parameters \( (\alpha, \beta, \gamma, \mu, \rho) \) as under constant returns to scale and, in addition, that \( \beta = 0.7 \), we solve for the numerical values of \( (L^{*}, c^{*}, \text{and } K^{*}) \):

\[ L^{*} = 758 \quad c^{*} = 1.86 \quad K^{*} = 13.05 \]

The long-run population level, per capita consumption, and capital stock are all somewhat lower under decreasing than under constant returns to scale. This result is general if the value of \( \alpha \) under constant returns to scale is the same as under decreasing returns to scale.\(^7\)

**CONCLUSIONS**

This paper has considered a social welfare function which incorporates the size of population in addition to per capita consumption as an argument to investigate its implications for optimal population policy. A modified Maudsley Rule has been derived to serve as a rule for optimal population growth. It has been shown that an optimal growth rate of population \( (\alpha < F'(K^{*})) \) exists and that in our model the steady state social return on investment equals the sum of the optimum population growth rate and the social rate of time preference \( (F'(K^{*}) - n + \rho) \). Based on these results, it is concluded that the application of the utility criterion to the analysis of dynamic optimum population unites the two divergent approaches that are captured by the per capita utility criterion and the total utility criterion, respectively. Specifically, the modified Golden Rule of the per capita utility approach and our version of the Maudsley Rule of the total utility approach are present in the same model. It should be noted that the derivation of these results requires discounting.

This study has argued that the externalities of population density are an important aspect of the quality of life. Economic growth involves capital accumulation, often accompanied by technological improvements, a process through which a higher optimum population can be maintained. On the other hand, environmental benefits such as open spaces and a pollution-free environment, important aspects of the quality of life, are likely to be highly income elastic and will tend to contribute to a declining optimum population as economies grow and per capita incomes rise. An advantage of our approach is that there is no need in our model to specify a "welfare substitution level" of consumption \( c_{w} \) (at which \( u_c = 0 \)), below which additional population is not justified, because we can use ordinal utility. Such specification is necessary in total utility models, because they are predicated on cardinal utility. A possible extension of our model would involve the explicit consideration of technological change and of its effect on the optimal outcomes obtained here.

**NOTES**

2. In my dissertation, "Population Models of Optimal Economic Growth," this subject was thoroughly investigated.
3. See also Constantinides (1987).
4. I owe this observation to an anonymous referee.
5. See also Pritchford (1974, p. 46).
OPTIMAL POPULATION GROWTH

The optimal capital accumulation rule for this model is the modified Golden Rule. We proceed with the derivation. The equation describing the costate variable is:

\[ p_k = -\frac{\Delta H}{\Delta k} \]

or equivalently in terms of the current value Hamiltonian:

\[ p_k = \frac{\Delta H}{\Delta k} + \rho p_k \]

Through this latter formulation, time (t) is eliminated as an explicit variable. Therefore,

\[ -\frac{\Delta H}{\Delta k} = \Gamma(k) \]

and

\[ -\frac{\Delta H}{\Delta k} = \Gamma(k) - \rho \]

Equations (A.1) and (A.2) show the percentage rate of decline of discounted and undiscounted marginal utility per capita, respectively. However, in order to solve for the rule for optimal capital accumulation, we need the equation for the percentage change in (discounted) marginal utility. Differentiating both sides of the first order condition of equation (2.8) with respect to time

\[ \frac{d(x^u_k)}{dt} - \frac{d(L_k)}{dt} = \frac{\Delta H}{\Delta k} = p_k \]

Substituting for \[ \frac{\Delta H}{\Delta k} \] from equation (A.3)

\[ \frac{d(L_k)}{dt} = p_k \Delta H \]

REFERENCES

Arrow, K.J., and M. Kurz, (1970), Public Investment, the Rate of Return, and Optimal Fiscal Policy. Baltimore: Johns Hopkins Press.


Therefore,

\[
\frac{\pi_l}{\pi_k} = \gamma(k) - n
\]

(A.6)

More than 30% of the males 14–63 years of age who were interviewed for the University of Michigan "Panel Study of Income Dynamics" experienced at least one dismissal over the years 1969–76.\footnote{2} Their experience suggests that the impact of layoffs on the financial well-being of workers deserves study. The aim of this study is to examine the impact of layoffs on the rate of wage growth, particularly through the effect of layoff-induced losses of firm-specific human capital.

A layoff can influence the financial well-being of an individual in at least two ways. First, a laid off worker who does not immediately start a new job will incur a loss in earnings for the period in which he did not work. The problem of unemployment (and of under-employment) while searching for a new full-time job is well known. Secondly, when re-employment is found it may be at a lower hourly wage than the individual previously earned. The estimation of this second effect, i.e., the effect of layoffs on the rate of growth of hourly wages rates, is the focus of this paper.\footnote{3} An important aspect of any negative effects of a layoff is their length. If layoffs decrease the rate of wage growth, it is important to know whether the damage is permanent, or, if not, how long it takes an individual to recover. Further, it is relevant to know how the impact of a layoff depends on a worker's personal characteristics, e.g., age, education, race and current job tenure. Both of these issues are addressed in this paper.

The job search literature has detailed two mechanisms by which dismissals might depress wage growth.\footnote{4} Since some dismissals are fires for cause, prospective employers may suspect dismissed workers of being less productive than average. If so, such workers will face a lower wage distribution than comparable job-seekers who have not been laid off.

Dismissal also implies the loss of some specific on-the-job-training (OJT). The more specific OJT a worker has with the current firm, the larger his current wage relative to his alternative wage with other firms and the larger the potential decrease in log wage growth from a layoff.\footnote{6}

While recognizing the potential impact of both of these mechanisms, this paper focuses on the second one. A potential problem with measuring the effect of layoffs on the rate of wage growth is that job mobility may be correlated with unobserved differences among workers in firm-specific OJT investments since the incidence of mobility is expected to be negatively related to a worker's specific OJT. Negative correlation between layoffs and unobserved differences in OJT would cause spurious negative correlation between wage growth and layoffs, because layoff prone individuals tend to have low wage growth regardless of their frequency of job change. One hypothesis to be tested in this paper is that individual differences in specific

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