Random Cost Functions and Production Decisions

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INTRODUCTION

Various researchers on the economics of uncertainty have come up with models to explain the "overall" and "marginal" impacts of uncertainty on the level of output and input utilization of the firm. Some of these researchers obtain their results under the assumption that the only source of uncertainty is output price (e.g., Sandino [1971], Ish [1977], Batra and Uliah [1974], Hartman [1975], Katz [1983, 1985], Foolad [1986]). Others assume input prices are random while output price is given (e.g., Okunoshita [1977], Blair [1974], Pierak [1982], Stewart [1978]). Foolad [1985] explores the effects of uncertainty, under the assumption that all prices (input and output) are random. Many economists also compare various economic decisions made by the firm under uncertainty with their counterparts under certainty, where the sources of uncertainties are something other than the output or input prices. These sources are quite diverse and range from uncertain deliveries of inputs (Martin [1961]) and random flow of factor services (Ratti and Uliah [1985]) to some specific technological uncertainty (Macmillan and Holtzmann [1983]) or uncertainty in parameters of production functions (Feldstein [1971]).

None of those researchers examine the effect of an uncertain cost function on the optimal level of output and input utilization of the firm. This effect can readily be determined with the framework established in this paper. The significance of assuming a random cost function is in its ability to summarize various types of uncertainties into one random variable. For instance, many of the above-mentioned sources of uncertainties (either solely or in combination) may create randomness in cost function.

THE MODEL

Random Cost Function with Certain Output Price

Consider a competitive firm seeking to maximize the expected utility of profits, $E[U(y)]$, where $y$ is the profit, $p$ is the output price, $x$ is the output level, and $c(x, \mu)$ is the random cost function. The random element in cost function, $\mu$, could be generated by an uncertain technological relation between output and input factors (i.e., an uncertain production function), uncertain deliveries of inputs, random flow of factor services, or any combination of these sources of uncertainties. For any given level of output, $c(x, \mu)$ depends upon $\mu$ which has cumulative distribution function $F(\mu)$.

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Assume
\[
\frac{\partial_c(x, \mu)}{\partial_c} > 0, \quad \frac{\partial_c(x, \mu)}{\partial \mu} > 0, \quad \text{and} \quad \frac{\partial^2(x, \mu)}{\partial c \partial \mu} > 0.
\]

The last assumption is equivalent to the condition that, for all \( k \), the sign of the partial derivative of marginal cost with respect to \( \mu \) is positive, i.e., it is equivalent to \( \partial^2 C(x, \mu) / \partial \mu^2 > 0 \). Alternatively, this assumption can be rewritten as \( \partial^2 C(x, \mu) / \partial \mu^2 > 0 \) which is equivalent to the condition that, as total expected cost increases (as a result of an increase in \( c \)), the dispersion of total cost will increase. This is analogous to the "principle of increasing uncertainty" (PIU) introduced by Leland [1972]. (Hereafter, this assumption will be referred to as PIU.)

It is assumed that the firm is a price taker (\( p \) is given), that the decision concerning the volume of output must be made prior to the knowledge of the random parameter \( \mu \), and that the firm exhibits a strictly convex, continuous, and twice-differentiable utility function of profits, i.e., \( u'(\mu) > 0, u''(\mu) < 0 \).

The objective of the firm is
\[
\max E[u'(\mu)] = E[u(p - c_1 - c_2 - \varepsilon(\varepsilon, \mu))].
\]

The first and second order conditions for the maximization are
\[
E[u'(\varepsilon)(\mu - M/C(\mu, \mu))] = 0 \quad \text{at } \alpha^*
\]
and
\[
E[u'(\varepsilon)(\mu - M/C) - \frac{\partial M/C}{\partial \mu} u'(\varepsilon)] < 0 \quad \text{at } \alpha^*
\]
where \( M/C(\mu, \mu) = \partial C(\mu, \mu)/\partial c \) and \( \alpha^* \) is in the optimum level of output.

It can be easily seen from (3) that increasing marginal cost (for any given \( \mu \)) is sufficient, but not necessary, to satisfy the second-order condition.

Having set forth the basic assumptions of the model and having derived the optimality conditions, it is now possible to examine the overall impact of uncertainty on the optimal levels of output and the quantities of inputs. More specifically, the optimal output and input demands under random cost function (as defined above) may now be compared with their counterparts determined when the cost function is known with certainty to be equal to its expected value, by the following theorem.

**Theorem 1** Under uncertainty, where the cost function is random for any given \( \mu \), the risk-averse competitive firm produces lower output than in the case where the cost function is known with certainty to be equal to \( E[C(x, \mu)] \).

**Proof**
(i) Following Leland [1972], define \( \mu(x) \) such that \( E[M/C(\mu, \mu)] = M/C(\mu, \mu) \).
(ii) Given the PIU and \( u''(\varepsilon) < 0 \), we will have the following: \( u'(\varepsilon) \leq M/C(\mu, \mu) \) and \( u'(\varepsilon) \geq M/C(\mu, \mu) \) where \( \varepsilon \) and \( \mu(x) \) is the level of profit for (given \( \varepsilon \)) evaluated at \( \mu(x) \).
(iii) From (ii) it follows that
\[
[u'(\varepsilon) - u''(\varepsilon)](M/C(\mu, \mu) - M/C(\mu, \mu)) \geq 0
\]
for all \( \mu \), and
\[
E[u'(\varepsilon)(M/C(\mu, \mu) - M/C(\mu, \mu))] = E[u'(\varepsilon)MC(\mu, \mu) - M/C(\mu, \mu)E[u'(\varepsilon)]] > 0
\]
(iv) From the first-order condition (2), we have
\[
E[u'(\varepsilon)MC(\mu, \mu)] = pE[u'(\varepsilon)].
\]

Substituting (6) into (5) results in
\[
pE[u'(\varepsilon)] - MC(\varepsilon, \mu)E[u'(\varepsilon)] = [p - MC(\varepsilon, \mu)]E[u'(\varepsilon)] > 0.
\]

Therefore, because \( E[u'(\varepsilon)] > 0 \), we have
\[
p > MC(\varepsilon, \mu) = E[MC(\varepsilon, \mu)]
\]
which means that the optimal level of output under uncertainty will be decided at a point where the output price is greater than the expected marginal cost.

(v) When the cost function is given to the firm as \( C(\mu, \mu) \) with certainty, the profit function is non-random. Therefore, maximizing the profit also maximizes the expected utility of profit. Thus, given a certain cost function, the firm will select \( x = x^* \), such as to satisfy the following first and second order conditions:
\[
\begin{align*}
\frac{\partial u}{\partial \mu} &= p - MC(\mu, \mu) = 0 \quad \text{at } \alpha^*, \quad \text{i.e.,} \quad p - MC(\mu, \mu) = E[MC(\mu, \mu)] \text{ at } \alpha^*, \quad \text{at } x^* \\
\frac{\partial^2 u}{\partial \mu^2} &= \frac{\partial MC(\mu, \mu)}{\partial \mu} < 0 \quad \text{at } \alpha^*, \quad \text{i.e.,} \quad \frac{\partial MC(\mu, \mu)}{\partial \mu} > 0
\end{align*}
\]

(8) Comparing (8) and (9) clearly illustrates that, given increasing marginal cost with respect to \( x \) (for any \( \mu \)), \( x^* < x^* \).

The implication of smaller output resulting from the presence of uncertainty is the lower usage of all inputs that are not inferior. This means that, in a two factor model, firm's demand for at least one of the inputs must decrease in order to decrease the output level and, if the production is so-called "well-behaved" (i.e., \( f_{c, i} > 0 \), demand for all inputs decreases.

**Random Cost Function and Output Price**

The results of the previous section hold, even if the assumption of the given output price is relaxed, i.e., if it is assumed that the decision concerning the volume of output must be made prior to the knowledge of the market price.

To see this, consider the same firm and assumptions as above, except now assume that the output price is a random variable having subjective cumulative probability distribution \( F(p) \), with the expected value \( E(p) = \hat{p} \) and variance \( var(p) = \sigma^2 \). To this case, the objective function of the firm, the first and second order conditions for maximum remains the same (as in (1), (2), and (3), respectively). However, equation (6) can be rewritten as follows:
\[
\begin{align*}
\frac{\partial E[u'(\varepsilon)]}{\partial \mu} &= \frac{\partial}{\partial \mu} E[u'(\varepsilon)] = E[u'(\varepsilon)MC(\varepsilon, \mu) + \frac{\partial}{\partial \mu} E[u'(\varepsilon)] + \frac{\partial}{\partial \mu} E[u'(\varepsilon)MC(\varepsilon, \mu)] + \frac{\partial}{\partial \mu} E[u'(\varepsilon)]
\end{align*}
\]

Rearranging (6) and dividing through by \( E[u'(\varepsilon)] \), we get
\[
\begin{align*}
\hat{p} - E(M/C) &= \frac{E[u'(\varepsilon)MC(\varepsilon, \mu)]}{E[u'(\varepsilon)]} - \frac{\text{cov}(u'(\varepsilon), M/C)}{E[u'(\varepsilon)]} \equiv \frac{\text{cov}(u'(\varepsilon), p)}{E[u'(\varepsilon)]}
\end{align*}
\]

(11)

We know that
\[
\frac{\partial u'(\varepsilon)}{\partial \mu} = u''(\varepsilon) > 0, \quad \frac{\partial u'(\varepsilon)}{\partial \mu} = u''(\varepsilon) \quad \text{and} \quad \frac{\partial M/C}{\partial \mu} > 0,
\]
by assumption. As a result, \( \text{cov}(u'(\varepsilon), M/C) > 0 \). We also know that
\[
\frac{\partial u'(\varepsilon)}{\partial \mu} = u''(\varepsilon) \sigma^2 > 0,
\]
and hence, \( \text{cov}(u'(\varepsilon), p) < 0 \). This means that the right-hand side of (11) is positive, resulting in
\[
\hat{p} > E(M/C) \quad \text{at } x^*.
\]
which is the same condition as (8) and therefore the same conclusion may be drawn as in Theorem 1. To elaborate on this point, note that if the output price and the cost function are given to the firm with certainty, as $P$ and $C(x)$ respectively, the profit function becomes non-random. Therefore, the firm will select $x = x_s$ such as to satisfy

$$P - MC(x_s) = E[MC(x)],$$

Given increasing marginal cost with respect to $x$ for any $x$, comparing (8) and (9) shows that $x_s > x$. Therefore, Theorem 1 can be restated as follows:

**Theorem 2** Under uncertainty, where the output price and the cost function are random, the risk averse competitive firm produces lower output than in the case where the output price and the cost function are known with certainty.

Our argument regarding the utilization of input factors would be exactly the same as it was for the case of given output price and we will obtain the same conclusion as was reached above.

**REFERENCES**


**INTRODUCTION**

Modern business cycle theory contends that observed values of saving, consumption, and labor supply represent the outcomes of dynamic optimization by economic agents. As a result, the theory implies that agents are able to substitute current saving and consumption for future saving and consumption, and current labor supply for future labor supply when they believe that such exchanges will be advantageous. This idea is known as the intertemporal substitution hypothesis (ISH). The principle that underlies this theory is that in the cost of their expected future utility in an uncertain environment, individuals must keep current decisions based on the parameters of expected future distributions. Empirical tests of the ISH have typically examined the relationship between current behavior and the means of expected future distributions. As a rule, these tests have not supported the ISH. This paper extends the ISH literature by examining the extent to which personal saving is subject to intertemporal substitution either as a result of changes in the mean of the expected future income distribution or as a result of changes in the variance of the expected future income distribution, other things paribus.

The idea that personal saving may be responsive to changes in the variance of the future income distribution is not new, having been discussed previously in separate papers by Hayne E. Leland (1968) and Agnar Sandmo (1970). A principle conclusion that emerges from both of these papers is that under reasonable assumptions “increased uncertainty about future income leads to more saving” (Sandmo, p. 556). Direct empirical tests of this hypothesis have been scarce, due to the difficulties involved in measuring the uncertainty associated with future income. This paper attempts to resolve that problem by employing Robert F. Engle’s (1982, 1983) model of autoregressive conditional heteroskedasticity (ARCH) to estimate the anticipated noise and variance of the future income distribution directly. The variance is used as a proxy for the level of expected future income uncertainty in estimated saving functions for the United States. While the estimates indicate that changes in the mean of the expected future income distribution have a negative effect on current saving, there is no support for the hypothesis that saving is subject to intertemporal substitution as a result of changes in the variance of the distribution.

The plan of the paper is as follows. Section II outlines the basic theory of saving under uncertainty, thus establishing the principal hypotheses to be tested. In Section III data and estimates of the aggregate saving function for the U.S. are presented. The paper is summarized and conclusions are drawn in Section IV.

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