A Note on the Bounded Solution of the Bilateral Monopoly Model

Chin W. Yang

Discussions of bilateral monopoly, i.e., a monopoly buyer (a monopolist) facing a monopoly seller, can be dated back to Cournot. As early as 1880, despite the divergent opinions on the solutions of the model, it is generally agreed that the characteristic of the bilateral model is its wide indeterminate solution range regarding the prices of the intermediate product. In a recent paper (1989), Bilir, Kaiser and Romano pointed out the apparent popularity of the incorrect solutions that exist in most microeconomic textbooks. However, they did not offer any new solutions to the bilateral monopoly model. In this note, we reexamine the model and propose a new solution that has a narrower price range than is suggested by Henderson and Quandt (1971, p. 247). Since Henderson and Quandt’s treatment on the topic is one of the few that are complete and correct (see Black, Kasten and Romano, 1989, p. 832), we shall present our analysis based on their model.

A REVISED SOLUTION

Consider a case of bilateral monopoly in which an intermediate product \( y \) (e.g., tobacco leaves) is sold to a single buyer who uses it to produce final product \( z \) (e.g., cigarettes). The buyer of the intermediate product sells \( z \) in a competitive market at a constant price \( P_z \) and a single seller of the intermediate product purchases raw input (or a combined input) \( x \) from a competitive market at a constant price \( p_x \) in order to produce \( y \). These production functions are shown below:

\[
\begin{align*}
& f: X \rightarrow Y \\
& g: Y_z \rightarrow Z
\end{align*}
\]

where \( f(x) = y \) is a functional mapping from input space \( X \) into intermediate output space \( Y \), and \( g(y) = Z \) as a functional mapping from \( Y \) into the final output space \( Z \). Note that we are only concerned with the interior of the intermediate product set \( Y = Y_x \cap Y_z \) so that the chain rule applies.

It is commonly assumed that either the single buyer or the single seller could dominate the market of the intermediate product. As a consequence, there may well be a tendency to move toward an "optimum" quantity of \( y \) that maximizes their joint profit:

\[
\begin{align*}
\text{Maximize} & = \pi_r + \pi_y = P_z \cdot P_y \cdot y + P_y \cdot P \cdot x \\
& \text{subject to } y \in Y, P \in R
\end{align*}
\]

\[
= P_z \cdot g(y) \cdot P_y \cdot y + P_y \cdot f'(y)
\]

* Clarion University of Pennsylvania, Clarion, Pennsylvania 16214
where \( \pi = \text{profit of the single buyer} \)
\[
\pi = \text{profit of the single seller}
\]
\[
P_y = \text{price of the intermediate product y}
\]
\[
P_x = \text{constant price of the final output x}
\]
\[
\gamma(y) = \text{constant price of the raw input x}
\]
\[
x = \text{raw input used in producing y}
\]

As is well-known in the bilateral monopoly model, differentiation of equation (3) equates a single buyer's value of marginal product of y to a single seller's marginal cost of
\[
P_y (\gamma(y)x) = \gamma(x)P_y
\]

The first order condition of equation (4) is the same as that derived from the competitive solution; hence, the optimality under collusion is identical to that of a competitive industry (see Henderson and Quandt, 1971). However, the price of the intermediate product y cannot be determined without some qualifications. It is evident from equation (3) that \(P_y\) is unbounded as long as \(n_x\) and \(n_y\) are unbounded, i.e., if the profit of the single buyer or seller can assume any negative value, then \(P_y\) can be any real number between zero and infinity. In this first bound on the solution (monotonicity to the bilateral model. However, it is unreasonable to assume that the intermediate product price \(P_y\) is no greater than \(P\) or the price that corresponds to zero profit for the single buyer \(\gamma(x) = 0\). On the other hand, one may set a lower limit on \(P_y\) such that the price of the intermediate product is no less than \(P\), i.e., the price that makes single seller's profit zero \((\gamma(x) = 0)\).

As is shown by Henderson and Quandt (1971, p. 247), \(P_y\) is bounded on one end by the zero profit for the single buyer, i.e., \(\gamma(x) = P_y + P_x = 0\) (or \(P_y = -P_x\)) for \(y = y^*\) as \(x = 0\) and \(n_x\) is maximized; and on the other end by the zero profit for the single seller, i.e., \(x = P_y - x = 0\) (or \(P_y = n_x\)) for \(y = y^*\) as \(x = 0\) and \(n_x\) is minimized. While this bound on the solution is narrower than the first bound, it is desirable to further narrow the solution bound. Henderson and Quandt (1971, p. 247) suggest that the single buyer can do no worse than the monopoly solution and that the single seller can do no worse than the monopoly solution. In terms of the mathematical programming model we have:

Maximize \(x = n_x + n_y\)

subject to \(P_x y + P_y x = n_y\)

\[
(\gamma(y)x) = \gamma(x)P_y
\]

where \(n_x\) = profit level of the single buyer under the monopoly market
\(n_y\) = profit level of the single seller under the monopoly market

Hence, from equations (6) and (7), the solution under the monopoly market bound suggested by Henderson and Quandt is

\[
(n_x + n_y) \leq P_x \leq (n_x + n_y)
\]

with \(y\) being evaluated at \(y^*\), i.e., optimum intermediate output in the joint profit problem.

The monopoly solution can be obtained from the following optimization problem:

Maximize \(x = n_x + n_y\)

subject to \(P_x y + P_y x = n_y\)

\[
\gamma(y)x = \gamma(x)P_y
\]

Constraint (10) states that price of the intermediate product set by the single seller equals its value of marginal product. Similarly, the monopoly solution can be obtained from the following optimization problem:
A GOAL PROGRAMMING APPROACH TO THE BILATERAL MODEL

As can be compared from the previous models, the profit under the joint profit maximizing model ($21,870) is greater than the sum of the profits under either monopoly ($540 + 12,150 = 17,500) or monopsony ($12,225 + 3,075 = 15,300) market. Hence, there is an incentive for the single buyer to move as close as possible to the profit level under the monopoly market ($5,150). Similarly, there is an incentive for the single seller to move as close as possible to the profit level under the monopoly market ($21,870). Obviously, both goals cannot be achieved simultaneously. In this case some kind of compromises are needed for the bilateral model to have a unique solution of the intermediate product price. Within this framework, we reformulate the bilateral monopoly problem into the following goal programming model:

Minimize \( a \Delta \alpha + b \Delta \gamma \) \[19\]
subject to \( \alpha_s + \Delta \alpha_s = \alpha_s^* \) \[20\]
\( \gamma_s + \Delta \gamma_s = \gamma_s^* \) \[21\]
\( \alpha_s + \gamma_s = \pi \) \[22\]
\( \alpha_s \geq 0 \) \[23\]
where \( \alpha_s \) = actual profit level of the single buyer in the bilateral monopoly model
\( \gamma_s \) = actual profit level of the single seller in the bilateral monopoly model
\( \Delta \alpha_s \) = deviation variable (underachievement) that represents the difference between the actual profit level of the single buyer (or \( \alpha_s \)) and the profit level under the monopoly market (or \( \alpha_s^* \))
\( \Delta \gamma_s \) = deviation variable (overachievement) that represents the difference between the actual profit level of the single seller (or \( \gamma_s \)) and the profit level under the monopoly market (or \( \gamma_s^* \))

\( \alpha^* = \) total profit of the joint profit maximization problem

On one hand, we could assume \( \beta = 0 \), i.e., the objective function is reduced to minimizing \( \Delta \gamma_s \). As a result, the deviation variable for the single buyer would be zero, that is, \( \Delta \alpha_s = 0 \) and \( \gamma_s = \gamma_s^* = 18,225 \), \( \pi = 3645 \), and \( \Delta \gamma_s = 8505 \) (or 12,150 - 3,075) with the same value of \( \alpha_s \) and \( \gamma_s \) (Table 2). This is equivalent to the revised solution with \( P = 148.5 \). On the other hand, we assume that \( \alpha = 0 \), i.e., the objective function is reduced to minimizing \( \Delta \alpha_s \). As a result, the deviation variable for the single seller would be zero, that is, \( \Delta \gamma_s = 0 \) and \( \gamma_s = \gamma_s^* = 15,150 \), \( \pi = 9720 \), \( \Delta \alpha_s = 8505 \) (or 18,225 - 9,725). This is equivalent to the revised solution with \( P = 300.6 \). The value of the deviation variables (8505) represents the amount of revenue either the single seller or single buyer fails short of the goal if the market were monopoly or monopsony, respectively. If the market were a monopoly, the maximum revenue goal would be \( \alpha_s = 12,150 \) for the single seller; and \( \gamma_s = 18,225 \) for the single buyer if the market were a monopsony. The sum of both \( \alpha_s \) and \( \gamma_s \) exceeds the maximum joint profit under the coalition ($21,870) by $8,505. The solution bounds on the revised model are a special case of the goal programming model or equations [19] through [23] in which either single seller or single buyer takes up all the amount of the underachievement from the goal. However, it is not likely that a single party would shoulder all the burden. An alternative solution to the bilateral monopoly model would be to assume that both parties share equal burden, i.e., \( \Delta \alpha_s = \Delta \gamma_s \). The solution is reported in Table 2; and the intermediate product price can be uniquely determined (\( P = 227.23 \)) in this case.

BOUNDED BILATERAL MONOPOLY MODEL

If the bargaining power or other information (e.g., the tax bracket for two parties) is such that the deviation from the goal of the single buyer is functionally related to that of the single seller, i.e., \( \Delta \alpha_s = h (\Delta \gamma_s) \), the bilateral monopoly model can also have a unique solution. As is shown in Table 2, the intermediate product price \( P = 253.59 \) for \( \Delta \alpha_s = 2 \Delta \gamma_s \). That is, each dollar's worth of the deviation from the goal for the single seller equals two dollars' worth of the deviation for the single buyer.

The Henderson and Quandt solution can be similarly formulated into the goal programming model with an objective function to maximize the overachievement of the minimum goal as shown below:

Maximize \( a \Delta \alpha_s + b \Delta \gamma_s \) \[24\]
subject to \( \alpha_s = \alpha_s^* \) \[25\]
\( \gamma_s = \gamma_s^* \) \[26\]
\( \alpha_s + \gamma_s = \pi \) \[27\]
\( \Delta \alpha_s \geq 0 \) \[28\]
\( \Delta \gamma_s \geq 0 \)

where \( \alpha_s \) and \( \gamma_s \) are the actual profit levels of the single buyer and seller in the bilateral monopoly market respectively
\( \alpha_s^* \) = the single buyer's profit in the monopoly market or the minimum profit level for the single buyer
\( \gamma_s^* \) = the single seller's profit in the monopoly market or the minimum profit level for the single seller
\( \Delta \alpha_s \) = deviation variable (overachievement) of the single buyer from the minimum profit level (\( \alpha_s^* \))
\( \Delta \gamma_s \) = deviation variable (overachievement) of the single seller from the minimum profit level (\( \gamma_s^* \))

\( \alpha^* \) = total profit of the joint profit maximization problem

For \( \beta = 0 \) all deviations (overachievement) from the minimum profit level (i.e., 21,870 - 3,075 = $18,795) go to the single seller. Hence, seller's actual profit is 16,470 (or 13,432.5 + 3,037.5). This solution (see Table 2) corresponds to the upper solution bound of the Henderson and Quandt model with \( P = 386 \) (see Table 1). For \( \beta = 0 \) all the deviations go to the single buyer; hence, it corresponds to the lower solution bound of the Henderson and Quandt model with \( P = 137.25 \). If the deviations from the minimum goals are to be distributed equally between the single buyer and seller, the profit levels would be $12,182.25 and 9737.75 respectively and with a unique \( P = 261.62 \). It is to be noted that this solution is not equivalent to that of \( \Delta \alpha_s = \Delta \gamma_s \) of the revised model. If the bargaining positions or the tax bracket considerations suggest the relationship that \( \Delta \gamma_s = 0.5 \Delta \alpha_s \), a unique intermediate product price can be found (\( P = 303.99 \)). Again, it is to be noted that this solution is not equivalent to that of \( \Delta \alpha_s = \Delta \gamma_s \) of the revised model since they are based on different behavioral assumptions.

NOTES
[1] \( \Delta \) = assumes the same value as in the comparison solution.
[2] Tables 1 and 2 are available upon request from the author.
REFERENCES


Book Reviews

Conflict and Effective Demand in Economic Growth by Peter Skott. Cambridge: Cambridge University Press, 1989, 175 p. $34.50.

One exciting development during the last decade has been the attempt to synthesize Marxist and Keynesian macroeconomics by Stephen Marglin, Amril Bhandari, Lance Taylor, and others. The basic idea is to marry the Keynesian theory of effective demand, characterized by an independent investment function and a two-class saving function, with the Marxist theory that the distribution of income is rooted in class conflict over wages and prices.1 In this short, dense, valuable book, Peter Skott contributes thoughtful discussions of many aspects of this literature, as well as a fully specified model. In this review I will focus on two innovative elements of Skott's model: his unique method of injecting class conflict into the model through the "output expansion function," and his fruitful attention to the microfoundations of investment spending.

In the approach of Marglin, Taylor and others to the synthetic model, investment spending is written, in somewhat of a fashion, as a function of profitability and, as a representation of the accelerator, the degree of capacity utilization. In this type of model an increase in the profit share (a cut in real wages) can have different effects on growth, depending on the parameters of the model. If profitability effects on investment dominate, growth will accelerate. Marglin and Bhandari call this an "exhilaration" regime; perhaps it would be more descriptive to call it a "Marxian" regime since it exemplifies profitability-constrained growth. On the other hand, if the accelerator effect dominates, growth will slow down. Recall that a lower wage share will reduce consumer demand, given the assumption that workers' saving propensities is less than that of capitalists. This regime is sometimes called "stagnationist" although it might be more descriptive to call such a demand-constrained economy "Keynesian". We will see below that Skott's model displays a similar Jekyll and Hyde personality, but in a way which depends not on the parameter values so much as on parameter one changes.

One criticism of the aforementioned investment function is that it too ad hoc; another is that it is hard to see why a firm with excess capacity would want to expand its capital stock. A traditional answer to the second criticism is that firms want to meet the demand for their product; Skott makes this principle the microfoundation for his investment theory, thus answering the criticism as well. Firms invest in order to realize a desired rate of capacity utilization. The desired rate of capacity utilization is inversely related to the profit share, since more profitable markets deserve to be more heavily protected.

For the short run, out of steady state equilibrium, Skott derives the investment function used by Marglin, Taylor and others. This investment equation helps explain the time path of the economy, which Skott argues will be that of a limit cycle under plausible parameter values. But it does not play a role in the long run steady state models from which Skott extracts comparative equilibrium results about changes in the power of workers or capitalist firms. In the long run, the function describing desired utilization doubles as the investment function.

Skott injects class struggle into his model by positing a short run output expansion function. Firms adjust the growth rate of production in response to changes in profitability (rather than in changes in inventories as in the textbook Keynesian model), but adjustment is costly. As employment increases, the costs of adjustment increase because workers are more favorably situated to interfere with managerial prerogatives. The growth rate of output is thus also inversely related to the employment rate.

Through this mechanism, Skott exploits the turning points in his limit cycle. The basic instability of the model is multiplier-accelerator instability, but class conflict contains the cycle at its extremes. As employment rises, for example, it causes firms to slow the growth rate of production as they move along their output expansion function. Capacity utilization declines, and this in turn reduces investment. Lower investment reduces the profit share through the familiar Keynes-Kaleckian mechanism, feeding back to still less investment. Soon a slump is underway, with employment falling. The recovery follows similar logic.