Equities in The Keynesian Model

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1. Introduction

In modern Keynesian macro models, it is conventional to consider only two financial assets: money and bonds. Except for recent papers by Sargent-Wallace and Tobin (1969), the literature on Keynesian models with unemployment ignores equities. A survey of texts on Keynesian macro analysis turns up the surprising fact that no one has included equity in their models. If equities are ignored in a Keynesian model, it follows that equities are in fact implicitly being handled in one of two possible ways: either they are simply assumed not to exist; or they are assumed to be perfectly substitutable with bonds, and are thus lumped in with bonds. Sargent and Wallace explicitly postulate the latter. They say (p. 474):

“The government has . . . liabilities the outstanding stock of money M and the outstanding stock of interest-bearing debt in the form of call loans B . . . We assume that consumers view call loans and equities as perfect substitutes when their yields are equal. Thus, in effect, reduce the number of paper-earning assets to one, the common yield on which is denoted r.”

The purpose of this paper is to extend the conventional short-run Keynesian model to allow for equities as a third asset, an asset which is imperfectly substitutable with bonds. We shall then compare this extended Keynesian model with two special cases: the case where equities are assumed to be perfect substitutes with bonds; and the even more special case where it is assumed that firms only issue bonds, and that equities do not exist. One extended Keynesian model will be constructed in the following section of the paper.

2. The Three Asset Keynesian Model

We assume that the typical firm produces a single output with two inputs, labor and capital services. The supply of capital services is fixed and is equal to a constant (assumed to be one) times the stock of capital. The production function is

\[ Y = F(N, K) \]  

where \( Y \) is output, \( N \) is employment, and \( K \) is capital services. The firm is assumed to maximize real profits subject to (1). The definition of real profits is

\[ \phi = Y - wN - rK \]  

where \( \phi \) is real profits, \( N \) is the money wage, \( r \) is the real cost of capital, and \( P \) is the money price of output.

Since \( K \) is fixed, the firm has only one decision variable in the short run, the demand for labor, \( N^* \); hence, there is one necessary condition for profit maximization, \( Y = wP \), or

\[ N^* = \frac{Y}{P} \]  

We assume throughout the paper that \( w \) is exogenous and that \( Y^* = N^* \) is less than the supply of labor.

Next consider the markets for assets. We assume that the real quantities of money, government bonds, and equities, that wealth owners desire to hold in their portfolios depend on the rates of return on these assets, the aggregate level of real income, and real wealth. We assume that the assets are gross substitutes, that is, that an increase in the rate of return on an asset will increase the demand for that asset at the expense of the others. Also, we assume that an increase in real wealth will increase the demand for all three assets; and that an increase in real income will raise the demand for money for transactions purposes at the expense of the other assets. We write these assumptions as follows:

3Sargent and Wallace argue that “the absence of a market for stocks of capital is the distinctive feature of this model.” (p. 483). It is the absence of such a market which permits discrepancies to exist in any time period between the marginal product of capital and the real cost of capital. Firms are assumed to be unable to adjust their capital stocks instantly because of high transaction costs, which include such things as brokerage fees, search costs, demutualizing and reutilization costs, bid-ask spread, and so on.

4See the excellent general equilibrium portfolio theory developed in Felsky-Sludski, (Ch. 3). We follow their approach.
The nominal rate of return on equities, \( \rho \), is defined as nominal dividends plus anticipated capital gains per dollar of equity; that is

\[
\rho = \frac{P^E - B^E}{P^E} + \gamma
\]

where \( \gamma \) is the expected capital gain per dollar of equity. Since equities are perceived as relatively inflation-proof, we assume that \( \gamma \) is an increasing function of the anticipated rate of inflation, \( \eta \). It is easily demonstrated that the nominal dividends is an increasing function of \( P \) and a decreasing function of \( \eta \). Thus, we have that

\[
\rho = \frac{\partial_d P^E}{\partial_d P} + \gamma(\eta), \quad \partial_d P > 0, \quad \partial_d \eta < 0, \quad \gamma(\eta) > 0
\]

(11)

which can be written in general form as

\[
\rho = \rho(P, W, \sigma, \eta), \quad \partial_\sigma \rho > 0, \quad \partial_\eta \rho < 0, \quad \partial_\rho \eta > 0
\]

(11)

Substituting (7) and (11) into (8), (9) and (10), we may rewrite the equilibrium conditions for the asset markets as

\[
\begin{align*}
B^E &= \rho(P, W, \sigma, \eta, \rho, M^*, P^*, B^E), \\
\eta^E &= \eta(P, W, \sigma, \eta, M^*, P^*, B^E), \\
\beta^E &= \beta > 0, \quad \beta^E < 0, \quad \beta^E > 0, \quad \beta^E < 0
\end{align*}
\]

(12)

\[
\begin{align*}
\rho^E &= \rho, \quad \rho^E < 0, \quad \rho^E > 0, \quad \rho^E > 0, \quad \rho^E > 0
\end{align*}
\]

(13)

To see this, solve the production function for \( \eta \) given \( N \) as an increasing function of \( Y \). Substitute this function into the expression for nominal dividends. Then, compute the partial derivatives of nominal dividends with respect to \( P, W, \) and \( \eta \). The last will be zero since \( \rho^E = \rho^E \).

11For convenience we often write \( \omega \) and \( \omega^* \) as separate variables throughout the paper. But keep in mind that these two variables always appear as a product.

10For empirical support of this elasticity assumption concerning money, see Muth. This assumption does not affect our analytical results using the three asset model, but it does have implications for interest rate behavior in the two asset model.

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As for the investment demand function, we assume that the typical firm determines investment in accordance with the value of \( \rho^E \). It is, we assume that desired investment is an increasing function of the difference between the marginal product of capital, evaluated at the given level of the capital stock, and the real cost of capital. This can be written as

\[
I = l(\gamma) + I[P^E - (\rho - \eta)], \quad \gamma > 0
\]

(17)

where \( \gamma = \rho - \eta \).

Since the capital stock is assumed fixed, and \( P^E R > 0 \), it follows from the production function that \( P^E R \) is an increasing function of \( Y \). Thus, using (11) (17), we may rewrite (17) as

\[
I = l^E(Y, \rho, W, \sigma, \eta, \rho^*)
\]

(18)

Notice that \( \eta \) has an ambiguous partial effect on \( I \), for a change in \( n \) has an indeterminate effect on the real cost of capital. This ambiguity will vanish in the two asset model.

We may now assemble three equations and write the complete Keynesian model with three assets. This model assumes that \( P^E R \) is the equilibrium variable for the goods market, so that \( Y \) is brought into equality with \( C + I + B \) by the movement of \( P \). Also, the money model asserts that the movement of the money market is rigid. Since one of the asset markets is redundant, we may delete one; we elect to omit the money market. We thus write the model as follows:

\[
Y = C + I^E(Y, P, b, \sigma, \rho, M^*, P^*, B^*)
\]

(19)

\[
\frac{\partial^E}{\partial^E} = \frac{\partial^E}{\partial^E} = \frac{\partial^E}{\partial^E} = \frac{\partial^E}{\partial^E}
\]

(20)

\[
Y = Y^E(Y, P, b, \sigma, \rho, M^*, P^*, B^*)
\]

(21)

\[
Y = \frac{\beta^E}{\gamma^E}, \quad \gamma^E = 0
\]

(22)

3. The Two Asset Keynesian Models

The above model assumes that bonds and equities are imperfect substitutes. If they are assumed to be perfect substitutes, then the portfolio equations will have to be altered; for, in the case, the nominal rates of return on bonds and equities must be equal in equilibrium. That is, if bonds and equities are assumed to be perfect substitutes, then we must add one more equilibrium condition to the above model, \( p^* + b^* = p^* \). It follows that the bond market and the equities market are no longer independent markets, for they are now constrained to determine the same rate of return. We must therefore add together the supplies and demands for stocks and equities, giving us a demand and a supply of "bond-equivalents" which yields a common rate of return, \( b^* + p^* \). One upshot of setting \( b^* = p^* \) and merging the bond and equities markets, is that \( s^* \) and \( s^* \) are eliminated as independent variables from the portfolio equations. We now proceed to transform the three asset model into a two asset model by assuming that bonds and equities are perfect substitutes.

10See Sargent-Wallace (p. 471).
BEGINNING WITH (8), (9), AND (10), SUBSTITUTE FOR \( \rho \) AND ADD TOGETHER THE SUPPLIES AND DEMANDS FOR EQUITIES AND BONDS. THEN, SUBSTITUTE EQUATION (11) AND SOLVE FOR \( \lambda \), WHICH GIVES

\[
\lambda = \frac{1}{\gamma} \left( b - \frac{\lambda - \gamma(0)}{\beta} \right)
\]

USING (23) TO ELIMINATE \( \lambda \) FROM (8), (9), AND (10), WE CAN WRITE THE EQUILIBRIUM CONDITIONS FOR THE ASSET MARKETS IN THE CASE OF PERFECT SUBSTITUTABILITY BETWEEN BONDS AND EQUITIES AS

\[
\psi(P, b, \gamma(0)) + \frac{b^*}{b} P = \psi(P, b, \gamma(0)) + \frac{b^*}{b} P
\]

THE EQUILIBRIUM CONDITION FOR THE BOND-EQUITIES MARKET, AND

\[
M^r = \left[ b, \psi(P, b, \gamma(0)) + \frac{b^*}{b} P \right]
\]

WHERE \( G_1 > 0, G_2 < 0, 0 < G_3 < 1, H_1 < 0, H_2 > 0, 0 < H_3 < 1 \). WE SHALL ASSUME THAT \( 0 < G_1, G_2, H_1, H_2 < 1 \).

BECAUSE THERE ARE NOW ONLY TWO ASSETS, THE EQUILIBRIUM CONDITIONS IN (24) AND (25) ARE NOT INDEPENDENT CONDITIONS, AND WE MAY DELETE EITHER OF THEM. WE CHOOSE TO OMIT THE BOND-EQUITIES MARKET—HERE REFERRED TO AS THE "BOND MARKET," FOLLOWING THE USUAL KEYNESIAN NORMATIVE nomenclature. RESTRICTING THE EQUILIBRIUM CONDITION FOR THE MONEY MARKET, WE HAVE

\[
M^m = \left[ b, \psi(P, b, \gamma(0)) + \frac{b^*}{b} P \right]
\]

WHERE \( G_1 > 0, G_2 < 0, 0 < G_3 < 1, H_1 < 0, H_2 > 0, 0 < H_3 < 1 \). WE SHALL ASSUME THAT \( 0 < G_1, G_2, H_1, H_2 < 1 \).

TO ISOLATE THE EFFECT OF IMPERFECT SUBSTITUTABILITY ON THE MONEY DEMAND FUNCTION, IT IS NECESSARY TO ELIMINATE \( \lambda \) FROM (14) IN ORDER TO MAKE (14) DIRECTLY COMPARABLE WITH (26). TO DO THIS, SOLVE (13) FOR \( \lambda \), AND SUBSTITUTE THIS EXPRESSION INTO THE MONEY DEMAND FUNCTION.
As with the money demand function, we find in the case of no equities that $n$ and $w$ play no role, and that the effect of $p$ becomes unambiguous. The reasons for these results are the same as for (28).

Finally, let us compare the investment demand function in the three models. In both two asset models the real cost of capital is $b - n$, hence if investment is assumed to be an increasing function of $bx$, we have

$$I = I(Y, b, n), \quad I_x > 0, I_b < 0, I_n > 0 \quad (32)$$

which is the investment demand function in both money-bonds models. To compare this function with (18), solve (13) once again for $n_x$, and substitute this solution into (18), which yields

$$I = I_n(Y, b, n, W, B, M), \quad n_x = 0, n_b > 0, n_n > 0 \quad (33)$$

which is another version of the investment demand function in the three asset model, one which is comparable with (32). Clearly, there are numerous differences between (32) and (33). The explanations for these differences are straightforward and we leave them for the reader.

$$\begin{align*}
Y &= C(Y, P, b, n, W, B, M) + I(Y, b, n) \\
M_f &= M(Y, P, b, n, W, B, M) \quad (34)
\end{align*}$$

where the bond market has been deleted; and where $Y, P, b,$ and $n$ are the endogenous variables.

The complete Keynesian model in the money-bonds case, where equities are assumed not to exist, is

$$Y = C(Y, P, b, n, W, B, M) + I(Y, b, n) \quad (35)$$

$$\begin{align*}
M_f &= M(Y, P, b, n, W, B, M) \\
Y &= Y_p \quad \text{(22)}
\end{align*}$$

where the bond market has been deleted; and where $Y, P, b,$ and $n$ are the endogenous variables.

### 4. Comparative Static Properties

In this section we investigate the comparative static properties of the three models. We explore the equilibrium response of these models to changes in the exogenous variables $n, W, B, M_f$, and $P$, as well as to an autonomous change in aggregate demand.

Substituting (22) into (19) through (21), we may write the Jacobian matrix of the three asset system as

$$\begin{align*}
\begin{bmatrix}
Y' \\
M_f' \\
B'
\end{bmatrix} &= \begin{bmatrix}
C_Y + I_x + I_y \\
C_P + I_b \\
C_n + I_n
\end{bmatrix} \\
&+ \begin{bmatrix}
\alpha V' \\
\beta V' \\
\gamma W' + \delta M_f'
\end{bmatrix}
\begin{bmatrix}
\delta P \\
\beta P \\
\alpha P
\end{bmatrix}
\end{align*}$$

where the sign pattern of this matrix, assuming $C_Y + I_x < 1$, is

$$\begin{array}{ccc}
+ & + & + \\
- & - & -
\end{array}$$

We now assemble these two-asset results and write them in the case of assumed perfect substitutability between equities and bonds,

$$Y = C(Y, P, b, n, B, M) + I(Y, b, n) \quad (34)$$

Consider the ambiguous sign on the term in the first column, second row. This term is the response in the excess demand for equities to a change in $P$, allowing for the effect of $P$ on this excess demand via $Y$. In the paragraphs following (14) we assumed that a rise in $P$ given $Y$ would create an excess demand for equities. But, this rise in $P$ also increases $Y$, and the latter (see (13)) lowers the demand for equities. Thus, a rise in $P$—allowing for the effect of $Y$—has an indeterminate effect on the excess demand for equities. Also, in the paragraph following (14), we reasoned that a rise in $P$ given $Y$ will create an excess supply of bonds, hence, a rise in $P$—allowing for the effect of $Y$—will generate an excess supply of bonds. This reasoning accounts for the positive sign on the term in the first column and third row.

Employing the correspondence principle, we deduce that the sign of the determinant of this Jacobian matrix is negative. Also, we are able to deduce that the minor of the element in the first row, first column must be positive. This fact is proved by employing the monotonicity requirements which derive from the wealth constraint in (7). (A proof is given in the Appendix.) Using this information, we can deduce the qualitative response of the three asset complete Keynesian system to a variety of assumed changes in exogenous variables. These results appear in Table 1.

Next, consider the two asset Keynesian system assuming perfect substitutability between equities and bonds. Using (22) in (26) and (34), we see that the Jacobian matrix for this system is

$$\begin{align*}
-\frac{Y'}{P} &= \left(\frac{C_Y + I_x + I_y}{P}\right) + \frac{C_P + I_b}{P} \\
-\frac{M_f'}{P} &= \left(\frac{\alpha V' + \beta V' + \gamma W'}{P}\right) + \frac{\delta M_f'}{P}
\end{align*}$$

where the sign pattern is taken to be

$$\begin{array}{ccc}
++ & + & + \\
+ & + & +
\end{array}$$

We have made the following additional assumptions to obtain this sign pattern. First, we assume that the marginal propensity to spend is less than one, that is, $C_Y + I_x > 1$. Second, we assume that $C_P$, which is ambiguous in sign, is not of such a large positive magnitude that a rise in $P$, allowing for the induced increase in $Y$, will generate an excess demand for output.

3Recall that this reasoning hinges on the assumption that the wealth elasticity of each asset demand equals one. Dropping this assumption remains the sign in the lower left hand corner of the Jacobian matrix with a question mark. This situation has no effect on the comparative static results in the three asset model.
With this information, we derive the response of the two asset system with perfect substitutability to the exogenous changes assumed for the three asset system. The results are presented in the upper half of the cells in Table 2.

Finally, consider the two asset Keynesian model with no equities. Using (22) in (28) and (35), we easily write the Jacobian matrix of this system. This matrix is exactly the same as that of the other two asset model with the C and M functions replacing the C and M functions, respectively. Also, the sign pattern of this two asset system is taken to be the same as that of the above two asset system, for essentially the same reasons. The comparative static results for this model are recorded in the lower half of the cells in Table 2.

5. An Interpretation of the Comparative Static Results

As Tables 1 and 2 indicate, there are very few differences in the way the three asset model and the two asset model, with perfect substitutability, respond to the assumed disturbances. Comparing the two asset models, we find their behavior the same except for their reaction to autonomous changes in the anticipated inflation rate and the money wage rate. In this section, we shall explain the economic reasons for these results.

### Table 2

<table>
<thead>
<tr>
<th>Equilibrium Change in</th>
<th>Increase in Aggregate Demand in</th>
<th>Increase in</th>
<th>Increase in</th>
<th>Increase in</th>
<th>Increase in</th>
<th>Increase in</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Aggregate Demand</td>
<td>m^2</td>
<td>m</td>
<td>b^2</td>
<td>b</td>
<td>(\pi^2)</td>
</tr>
<tr>
<td>(\pi)</td>
<td>+ +</td>
<td>+ / -</td>
<td>+ / -</td>
<td>+ / -</td>
<td>0 / 0</td>
<td></td>
</tr>
<tr>
<td>(\pi)</td>
<td>+ 0</td>
<td>+ / 0</td>
<td>+ / 0</td>
<td>+ / 0</td>
<td>0 / 0</td>
<td></td>
</tr>
<tr>
<td>(b) or (\pi)</td>
<td>- / +</td>
<td>- / +</td>
<td>- / +</td>
<td>- / +</td>
<td>- / +</td>
<td></td>
</tr>
</tbody>
</table>

The signs in the upper half of the table indicate the response of the two asset system assuming perfect substitutability between equities and bonds; the signs in the lower half refer to the two asset system with no equities.

An autonomous increase in aggregate demand

In the two asset models, an autonomous rise in aggregate demand creates an excess demand for output at the initial equilibrium \(P\) and \(b\). Thus, equilibrium \(P\) and \(Y\) must rise. As \(P\) and \(Y\) rise, an excess supply of bonds is created, hence equilibrium \(b\) will rise. Consider now the response of the three asset system to this exogenous change. Both equilibria \(P\) and \(Y\) must rise in the simpler systems, but in this case the effect on the asset markets is ambiguous. The rise in \(P\) and \(Y\) will have an indeterminate effect on the excess demand for equities, hence on \(\pi\), and since \(\pi\) is an argument in the bond demand function, the response of the excess demand for bonds to the rise in \(P\) and \(Y\) is thus ambiguous. It follows that the response of the asset markets to an autonomous increase in aggregate demand is a priori indeterminate. Notice, however, that even in the three asset system, the impact of fiscal policy on \(P\), \(Y\), and \(\pi\) is in the usual direction.

Increases in the stocks of assets

In the two asset systems, the effects of changes in \(m^2\) and \(b^2\) on the system are well-known. A rise in \(m^2\) will raise aggregate demand at \(b\) and \(\pi\), both lowering \(b\) and \(\pi\), and directly increasing wealth; the resulting rise in \(P\) and \(Y\) will raise the demand for money (lower the bond demand) with the new equilibrium \(b\) either higher or lower than the initial equilibrium \(b\). If the wealth effect in the consumption function is small enough, it is easy to establish that a rise in \(m^2\) will lower equilibrium \(b\). A rise in \(b^2\) in the two asset system will have ambiguous effects on aggregate demand at a given \(Y\), for the rise in \(b^2\) will (via the wealth effect in the money demand function) raise \(b\) at a given \(Y\), reducing aggregate demand; at the same time, this rise in \(b^2\) (via the wealth effect in the consumption function) raises aggregate demand at a given \(Y\). Thus a rise in \(b^2\) will have indeterminate effects on equilibrium \(P\) and \(b\), but must raise equilibrium \(b\). The new equilibrium \(b\) cannot be lower than the initial equilibrium \(b\), for this would imply that equilibrium \(P\) and \(Y\) fell by an amount large enough to create the equilibrium fall in \(b\). But this is logically impossible because a lower \(b\) and a higher \(b\) imply that aggregate demand must exceed \(Y\), at the initial equilibrium \(Y\).

Now, consider an increase in the money supply in all three asset model. Aggregate demand at the initial equilibrium \(P\) must rise for three reasons. First, the rise in \(m^2\) directly increases real wealth. Second, the rise in \(m^2\) creates an excess demand for equities, which (at a given \(P\) and \(Y\)) raises \(\pi\), hence real wealth. Third, the rise in \(m^2\) creates an excess demand for bonds which (at a given \(P\) and \(Y\)) lowers \(b\), thus increasing aggregate demand. Since aggregate demand increases at the initial \(P\) and \(Y\), it follows that \(P\) and \(Y\) must move to higher equilibrium values. As \(P\) and \(Y\) move up, however, the asset markets are affected ambiguously, thus the equilibrium response of \(\pi\) and \(b\) are indeterminate.

Next, consider an increase in \(b^2\) in the three asset model. This exogenous change also exerts three influences on aggregate demand at the initial equilibrium \(Y\), but unlike the case of an increase in \(m^2\), these effects are not all in the same direction. First, the rise in \(b^2\) raises real wealth, ceteris paribus. Second, the rise in \(b^2\) creates an excess demand for equities (at a given \(P\) and \(Y\)) and an increase in \(\pi\) that lowers aggregate demand. It follows that a rise in \(b^2\) may raise or lower aggregate demand at the initial equilibrium \(P\) and \(Y\), hence the equilibrium changes in \(P\) and \(Y\) are indeterminate. Clearly if wealth effects are deleted from the consumption function, a rise in \(b^2\) will lower \(P\) and \(Y\). Regardless of the equilibrium changes in \(P\) and \(Y\), the equilibrium responses of \(b\) and \(\pi\) are indeterminate, since the asset markets respond ambiguously to changes in \(P\) and \(Y\). As to an increase in \(\pi^2\), it is clear that it can only lower \(\pi\) proportionately in both models. The reason for this can be seen in (7) and (11) where \(\pi\) and \(\pi^2\) always enter only as a product.

An autonomous increase in the anticipated rate of inflation

In the two asset model with perfect substitutability, an exogenous rise in the expected inflation rate will affect aggregate demand at a given \(Y\) in three ways. First, the assumed rise in \(\pi\) implies an increase in real wealth which raises consumption demand at a given \(Y\). Second, the rise in \(\pi\) increases the expected return of capital given \(Y\) which both raises \(b\) and \(\pi\), hence real wealth. Third, the rise in \(\pi\) raises the expected return of capital given \(Y\) which increases investment demand. The upshot of these effects is that a rise in \(\pi\) may raise or lower aggregate demand at a given \(Y\); consequently, the equilibrium response of \(P\) and \(Y\) to the rise in \(\pi\) is ambiguous. In the two asset model with no equities, only the last effect is operative for there are no wealth effects at given \(P\), \(Y\), and \(\pi\); hence, a rise in \(\pi\) will increase equilibrium \(P\) and \(Y\).

It is not difficult to establish that in the two asset model, a rise in \(\pi\) must raise equilibrium \(b\). To see this, imagine that \(b\) is faller and \(P\) and \(Y\) have risen. These hypothetical changes

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The table above shows the response of the two asset system to selected exogenous changes.
imply an excess demand for money. Conversely, imagine that $b$ has fallen, and that $P$ and $Y$ also have fallen. These hypothetical changes imply an excess demand for goods. It follows that equilibrium $b$ must rise in response to an increase in $\eta$ in both two asset models, regardless of how equilibrium $P$ and $Y$ change.

It is also possible to deduce the equilibrium change in $b - \eta$ in the two asset models, in response to a rise in $\eta$. In the no equation case, a rise in $P$ and $Y$ imply an excess supply of goods at the initial $b - \eta$; hence, the new equilibrium in the goods market requires that the percentage increase in $b$ be less than the percentage increase in $\eta$. By similar reasoning, in the perfect substitutability case, if equilibrium $P$ and $Y$ rise, then equilibrium $b - \eta$ must fall and vice versa.

Consider now the impact of a rise in $\eta$ using the three asset system. This rise affects aggregate demand at given levels of $P$ and $Y$ in three ways which are similar to those discussed in the two asset systems. First, the real cost of capital may rise or fall; ceteris paribus--hence aggregate demand may rise or fall. Second, the demand for equities rises because $\rho$ rises, increasing the given levels of $P$ and $Y$, which increases real wealth and hence aggregate demand. Third, an excess supply of bonds is created by the rise in $\eta$, hence $\rho$, which raises $b$ at given levels of $P$ and $Y$, which reduces aggregate demand. Hence, a rise in $\eta$ has ambiguous effects on aggregate demand at the initial $P$ and $Y$; and therefore the equilibrium response of $P$ and $Y$ to a change in $\eta$ is ambiguous, as are the equilibrium responses in $b$, $P$, and $Y$.

An autonomous increase in money wages

The equilibrium response of every variable in the three asset system to an exogenous change in the money wage is qualitatively indeterminate. The reason for this lies in the behavior of the asset markets. When a change in $P$ affects $P$ and $Y$, the excess demand for assets respond ambiguously, which implies that $P$ and $b$ move in an indeterminate fashion; hence, it is clear that nothing can be said a priori about the direction of response of any variables in the three asset system.

The two asset model with perfect substitutability also responds ambiguously to an autonomous change in the money wage. A rise in $W$ lowers real wealth and hence consumption demand at a given $Y$, but by reducing real wealth, the rise in $W$ also lowers the demand for money which lowers $b$ thus increasing aggregate demand at a given $Y$. It follows that equilibrium $P$, $Y$, and $b$ may rise or fall in response to an autonomous increase in $W$. The situation is quite different in the two asset model with no equities, for in that model a change in $W$ will have no real wealth effect at all. Thus, a rise in $W$ will lower $Y$, given $P$ and $b$, which will create an excess demand for goods and an excess supply of real money balances at the initial values of $P$ and $b$. Thus, $P$ will rise and $b$ will fall until equilibrium is restored in all markets. Can $P$ rise by the same proportion that $W$ did, implying no equilibrium change in $Y$? The answer is no; $P$ must rise by a smaller proportion than $W$. To see this imagine that $P$ did rise by the same percentage as $W$; if the elasticity of money demand with respect to wealth is one, this implies that equilibrium in the money market requires no equilibrium change in $b$, while equilibrium in the goods market requires an equilibrium fall in $b$ (to offset the negative wealth effect of the rise in $P$). Hence, a rise in $W$ will generate an equilibrium increase in the real wage, a fall in $Y$, and a fall in $b$.

6. Summary

Several interesting analytical results emerge when imperfect substitutability of bonds and equities is allowed for, and the two asset complete Keynesian models are thereby extended to a three asset model:

1. The money demand function is altered in the following ways by assuming imperfect substitutability. The partial derivatives of money demand with respect to real GNP, the anticipated inflation rate, and the money wage all become ambiguous. In the case of perfect substitutability, these partial derivative signs are $(+, +, +)$, respectively; while in the no equities case, these signs are $(+, 0, 0)$, respectively.

2. In the case of imperfect substitutability, the marginal propensity to consume becomes indeterminate, while in the two asset models the MPC has the usual positive sign.

3. The investment demand function is altered in a variety of ways by extending the models to three assets. The partial derivatives of investment with respect to real GNP and the anticipated inflation rate become ambiguous when imperfect substitutability is assumed; while in both two asset models, these partial derivative signs are both positive. Also, in the three asset case, the partial derivatives of investment demand with respect to the price level, the wage level, the stock of money, and the stock of government bonds have the signs $(+, +, +, +)$ respectively; whereas, in the two asset investment demand function, these arguments do not appear.

4. An autonomous rise in aggregate demand or in the supply of government bonds or in the anticipated inflation rate will raise the equilibrium rate of interest in the two asset models; while, in the three asset model, the interest rate responds in indeterminate fashion to these changes.

5. The main reason that the three asset system exhibits the ambiguities mentioned above lies in the behavior of the equities market and in the fact that this market interacts with the other markets. A crucial aspect of equity market behavior in this respect is that the demand for equities responds ambiguously both to GNP and to the price level.

6. An autonomous rise in the anticipated inflation rate and in the money wage rate has indeterminate effects on the price level, output, and employment in the three asset model as well as in the two asset model with perfect substitutability. In the no equities model, a rise in

the money wage rate will raise the price level and lower output and employment; whereas, a rise in the anticipated inflation rate will increase the price level, output, and employment.

7. Finally, it is noteworthy that the direction of effect which monetary and fiscal actions have on the real sector is not altered by extending the complete Keynesian models to three assets.

Appendix

We prove that the following principal minor of the Jacobian of the three asset system

$$\frac{\partial^2 S}{\partial P^2} \frac{\partial^2 S}{\partial P \partial Y} \frac{\partial^2 S}{\partial Y^2} - \frac{\partial^2 S}{\partial P \partial \eta}$$

is positive. Since there are only three assets, the following identity must hold (see (79)),

$$\frac{\partial^2 S}{\partial P^2} + \frac{\partial^2 S}{\partial P \partial Y} + \frac{\partial^2 S}{\partial Y^2} = 0$$

and

$$\frac{\partial^2 S}{\partial P \partial \eta} = 0$$

where the asset demand functions on the RHS are spelled out in (12) to (14). Computing the partial derivatives of (A.1) with respect to $\pi$ and $b$, we have

$$\frac{\partial^2 S}{\partial P \partial \eta} = 0$$

and

$$\frac{\partial^2 S}{\partial P \partial \eta} = 0$$

Since $M^2_2 > 0$ and $M_1^2 < 0$, it follows that

$$\frac{\partial^2 S}{\partial P^2} > \frac{\partial^2 S}{\partial Y^2}$$

and

$$\frac{\partial^2 S}{\partial P \partial \eta} > \frac{\partial^2 S}{\partial Y \partial \eta}$$

Hence, from (A.4) and (A.5), we can write
\[
\lambda \left( \frac{\mu \sigma^2}{P} - \frac{\sigma^2}{P} \right) = \beta_4^2 - \sigma_4^2 + \frac{\sigma^2}{P} \quad 0 < \lambda < 1 \quad (A.6)
\]

and
\[
\mu \left( \frac{\beta_4^2}{\sigma^2} + \frac{\mu \sigma^2}{P \sigma^2} \right) = -\frac{\sigma^2}{P} \quad 0 < \mu < 1 \quad (A.7)
\]

Substituting the left sides of (A.6) and (A.7) for \( \beta_4^2 \) and \( \sigma_4^2 \), respectively, in the above minor we can rewrite that minor as
\[
\left( \frac{\mu \sigma^2}{P} - \frac{\sigma^2}{P} \right) \left( \frac{\beta_4^2}{\sigma^2} + \frac{\mu \sigma^2}{P \sigma^2} \right) = \lambda \left( \frac{\mu \sigma^2}{P} - \frac{\sigma^2}{P} \right) \left( \frac{\beta_4^2}{\sigma^2} + \frac{\mu \sigma^2}{P \sigma^2} \right)
\]

Expanding the minor, we have
\[
(\lambda \mu - 1) \left( \frac{\beta_4^2}{\sigma^2} + \frac{\mu \sigma^2}{P \sigma^2} \right) \left( \frac{\sigma^2}{P} - \frac{\mu \sigma^2}{P} \right)
\]

which is positive since
\[
(\lambda \mu - 1) < 0
\]
\[
\left( \frac{\beta_4^2}{\sigma^2} + \frac{\mu \sigma^2}{P \sigma^2} \right) > 0
\]
\[
\left( \frac{\sigma^2}{P} - \frac{\mu \sigma^2}{P} \right) < 0
\]

Our proof is thus complete.

References


**"The Risk Free Rate, Competitive Markets, and the Capital Asset Pricing Mode”**

DENNIS E. LOGUE and MICHAEL A. SIMKOWITZ

Abstract

A number of empirical studies have concluded that the predictive content of the traditional capital asset pricing model is seriously hindered by its reliance upon the assumption that investors may lend or borrow all that they like at a single risk free rate. One important revision of the capital asset pricing model which does not use this assumption suggests that investors hold some linear combination of the market portfolio and a minimum variance portfolio which is uncorrelated with the market portfolio; it yields theoretical results which are much more consistent with the empirical findings. Using a single period state preference model, this paper argues that the former model is not consistent with a competitive equilibrium, whereas the latter is. This result derives directly from the fact that the assumption of a risk free rate exogenous to the determination of other market returns implies the existence of a monopoly security insurer: one who has access to some securities to which other investors do not have direct access.

I. Background

The traditional form of the capital asset pricing model holds that the equilibrium return on any security is a linear function of the return on a riskless asset and the return on a market portfolio. Using a single period state preference model, we argue that the notion of a competitive equilibrium for assets is inconsistent with the concept of a risk free rate which is necessarily exogenous to the competitive system which determines the returns on other capital assets. An exogenous risk free rate implies that there is a monopolistic insurer who has access to investment opportunities which are not available to other investors. If there were no such insurer, these special opportunities would be a part of the market portfolio and there would be no need to introduce the riskless asset in order to derive an equilibrium situation. Under these conditions, it is not surprising that direct empirical tests of the traditional asset pricing generally have failed, but that tests of the implications of the theory in particular tests of the efficiency of the market in adjusting to new information have met with outstanding success in demonstrating the competitive efficiency of the capital markets.

Recent direct empirical studies by Black,

