Eastern Economic Journal, Vol. IX, No. 1, January–March 1983

A Remark on the Graphical Exposition of Neo-Classical Two-Sector Growth Models

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I. Introduction

Graphical techniques for neo-classical two-sector growth models à la Uzawa (1961, 1963) have been widely used by several authors in the literature, including Professor Allen in his macro-text (1970), and Professor Johnson (1971, 1973), and his joint work with Professor Krauss (1975). In particular, the two techniques used in Professor Johnson's various works are known as the Lerner-Pearce diagram and the Edgeworth-Bowley box diagram. Although these methods have proved very useful in illustrating many important aspects of the two-sector economy in a static framework, they do not appear as helpful for a comparative-static or a dynamic analysis of economic growth. The simpler treatment for the latter subjects would seem to involve the presentation of the cross-section production functions relating per-capita output to capital–labor ratio. This alternative was in fact used in a well-known survey article on the theory of economic growth by Professors Hahn and Matthews (1964). Unfortunately, their graphical technique was not developed to its full capacity and therefore, was unable to give a complete geometrical version even for the simplest case of the classical saving assumption where workers consume all wages and capitalists save all profits (or using notation below, $s_w = 0$, and $s_r = 1$.) Using the Hahn and Matthews diagram, Professor Allen remarked in his text (1970, p. 278) that, in the case where $s_w = 0$ and $0 < s_r < 1$, the allocation of capital and labor between the two sectors, and the determination of the over-all capital–labor ratio as well as per-capita output, for a given level of wage–rental ratio, *cannot* be graphically represented in any simple way.

The major purpose of this note is to demonstrate that such a diagrammatic presentation can be made in a simple fashion for the case of $s_w = 0$ and $s_r = 1$, and that of $s_w = 0$ and $0 < s_r < 1$.[1] Moreover, in the course of discussion, we shall indicate how some of the well-known propositions on existence, stability, and uniqueness of equilibrium can be graphically shown. Thus, the present method should also serve as a useful alternative to the Johnson and Krauss technique mentioned above.

II. The Model: The Determination of Short-run Equilibrium

The production functions of consumption-goods sector (C), and capital-goods sector (I), which are assumed to be well-behaved and exhibit constant returns to scale, can be shown

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[1]The graphical presentation for the case of constant proportional saving assumption ($s_r = s_w = s$), or even for the more general situation where $s_w \neq s_r$ is a simple extension of the present technique. An analysis of these cases is available from the author upon request.

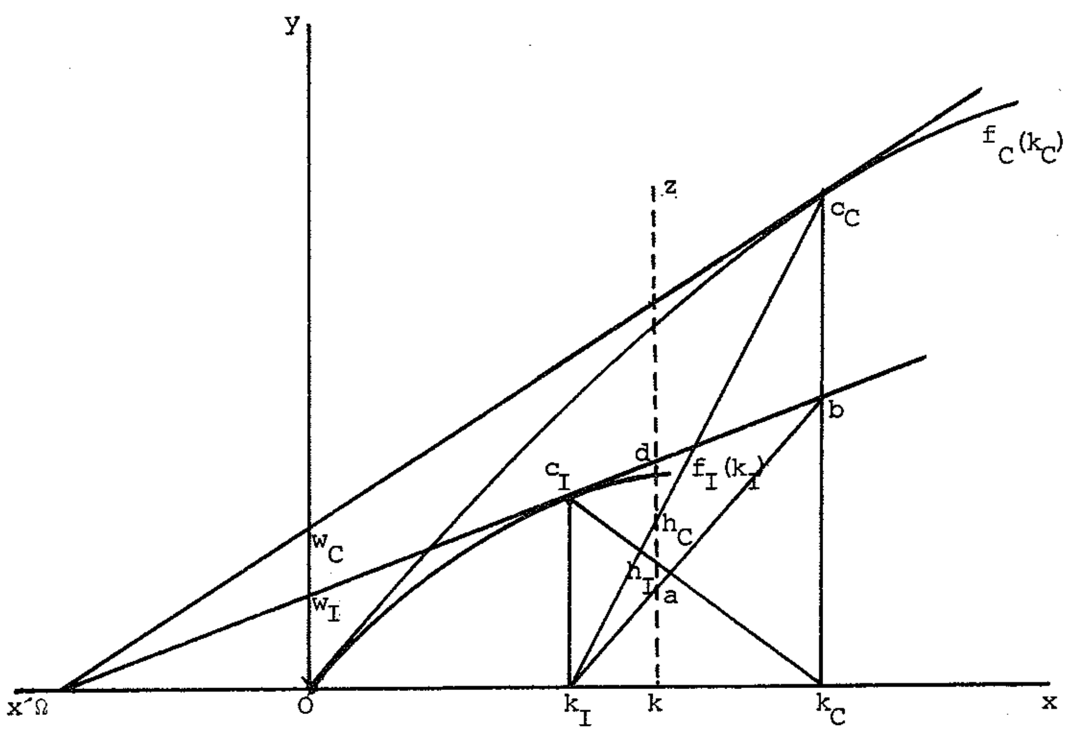

Figure 1

in Figure 1 where capital–labor ratios k's are on the horizontal axis and per-worker physical output y on the vertical one.[2] It is noted that the relative position of the two production curves $f_C(k_C)$ and $f_I(k_I)$ is immaterial as they merely reflect the scale or units used. It is also noted that the diagram is so constructed to satisfy the capital intensity hypothesis ($k_C > k_I$) as analyzed in Uzawa (1961) and (1963). Consider now the curve $f_I(k_I)$ which represents the per-worker production function for the capital goods sector. For the capital–labor ratio, k_I, the marginal product of capital is the slope of the curve $f_I(k_I)$ at c_I. The tangent to the curve $f_I(k_I)$ at c_I cuts the horizontal axis at Ω. It has been shown in the literature of one-sector growth models (Hahn-Matthews, Allen) that distance $\overline{o\Omega}$ indicates the corresponding wage–rental ratio ω, say. The efficient allocation of resources in a two-sector model would require that the wage–rental ratios between the two sectors are equal. In terms of Figure 1, the condition would require that the tangent to the curve $f_C(k_C)$ at c_C must cut the horizontal axis at the *same* point Ω. Thus, for a given wage–rental ratio ω, one can find a point Ω on ox' such that $\overline{o\Omega} = \omega$, from which two straight lines are drawn, one being tangent to the curve $f_C(k_C)$, another to $f_I(k_I)$. The first tangent helps to determine the level of k_C and the second to determine that of k_I. It can be easily seen that the higher the level of ω, the higher that of k_I and also that of k_C.

The determination of the over-all capital–labor ratio k would depend on different

[2] The notation used in this note is rather standard in the literature. See Uzawa (1963), for example.

assumptions on saving behaviors of the economy. Before turning to such a determination, let us see if per-capital levels of consumption goods ($y_C = Y_C/L$) and investment goods ($y_I = Y_I/L$) can be graphically shown in Figure 1. For a given level of k, draw the vertical line kz, perpendicular to ox. Two points c_I and k_C are connected by a straight line which in turn cuts kz at h_I. One obtains:

$$(\overline{kh_I}/\overline{k_Ic_I}) = (\overline{kk_C}/\overline{k_Ik_C}),$$

or

$$\overline{kh_I} = (\overline{kk_C}/\overline{k_Ik_C}) \cdot \overline{k_Ic_I},$$

where:[3]

$$\overline{k_Ic_I} \equiv f_I(k_I),$$

and

$$(\overline{kk_C}/\overline{k_Ik_C}) \equiv (k_C - k)/(k_C - k_I) = \ell_I.$$

Thus,

$$\overline{kh_I} \equiv \ell_I f_I(k_I) = y_I. \qquad (1)$$

Similarly,

$$\overline{kh_C} \equiv \ell_C f_C(k_C) = y_C \qquad (2)$$

where $\ell_C \equiv (\overline{k_Ik}/\overline{k_Ik_C})$.

To obtain the corresponding level of per-capita gross national income y, where $y = Y/L$, one has to introduce the notion of relative price. Let p denote the ratio of the price of the consumption good to that of the new capital good. This relative price p is represented in Figure 1 as the ratio $(\overline{ow_I}/\overline{ow_C})$, by noting that the money wage in terms of investment goods should be the same between the two sectors. (Note that $\overline{ow_I}$ and $\overline{ow_C}$ measure money wages in the I-sector and the C-sector, respectively, in terms of their own products.) By definition, gross national income y in terms of investment goods is:

$$y = y_I + p \cdot y_C \qquad (3)$$

where, from (2),

$$p \cdot y_C \equiv (\overline{ow_I}/\overline{ow_C}) \cdot (\overline{kh_C}) = (\overline{k_Cb}/\overline{k_Cc_C}) \cdot (\overline{kh_C}) = (\overline{ka}/\overline{kh_C}) \cdot (\overline{kh_C}) = \overline{ka}.$$

Furthermore, it is a simple matter to recognize that $\overline{ka} = \overline{h_Id}$. Hence, together with (1), equation (3) becomes:

$$y \equiv \overline{kh_I} + \overline{h_Id} = \overline{kd}.$$

In other words, for a given level of k, the distance $\overline{kd}$ represents the per-capita income y with the investment-goods component being shown by $\overline{kh_I}$ and consumption-goods by $\overline{h_Id}$. This geometric representation of per-capita income appears to be of interest and the present author is not aware of any graphical method in literature to deal with this kind of aggregation for a two-sector growth model.

The next question to be considered is the determination of the short-run equilibrium level of the over-all capital–labor ratio k. As already mentioned, this task would involve various types of assumption on saving behaviors. We demonstrate below only two simple cases.[4]

The Classical Saving Assumption ($s_r = 1, s_w = 0$)

This seems to represent the simplest case where all profits are saved and wages are consumed. Perfect competition would then dictate that $y_I = r \cdot k = f'_I(k_I) \cdot k$, and $p \cdot y_C = w$. In terms of our diagram, this would mean that the line oh_I must be parallel to the tangent line Ωc_I. The preceding analysis suggests an extremely easy way to determine graphically the level of capital–labor ratio for a given wage–rental ratio ω. Starting with a wage–rental ratio ω, one can determine k_I and k_C, then connecting points c_I and k_c by a

[3] Note that when both production factors are fully employed, $\ell_I k_I + \ell_C k_C = k$, and $\ell_I + \ell_C = 1$, hence $\ell_I = (k_C - k)/(k_C - k_I)$, and $\ell_C = (k - k_I)/(k_C - k_I)$.

[4] See footnote #1.

straight line. From the origin o, draw a line parallel to line Ωc_I which cuts line $c_I k_C$ at point h_I whose x-coordinate represents the corresponding capital–labor ratio k. This was in fact done in Figure 1 although the line oh_I is not explicitly drawn.

The Case Where $s_w = 0, 0 < s_r < 1$

Since capitalists now save only a positive proportion of profits, we have:

$$y_I = s_r \cdot r \cdot k = f_I'(k_I) \cdot s_r \cdot k \qquad (4)$$

where the second equality is due to the marginal productivity condition. The graphical determination of k so that (4) is satisfied can be made with the following steps the justifications of which to be given shortly. See Figure 2.

Step 1: From the origin o, draw a line parallel to line Ωc_I, which cuts line $c_I k_C$ at point e, and intersects with $k_C b$ at q. From point e, drop line eg which is perpendicular to the x-axis at g. The connect gq.

Step 2: On eg, find point m such that $(\overline{me}/\overline{ge}) = s_r$, then connect points o and m. The extension of om cuts gq at u whose x-coordinate represents the corresponding level of k.

These steps are carried out in Figure 2 where certain lines and curves from Figure 1 are omitted for a clearer presentation of the diagram. To justify the above steps, it is noted from the triangle okq' that the value of $f_I'(k_I) \cdot k$ can be presented by distance $\overline{kq}'$. It would follow immediately that distance $\overline{uq}'$ shows the value of $f_I'(k_I) \cdot s_r \cdot k$ since

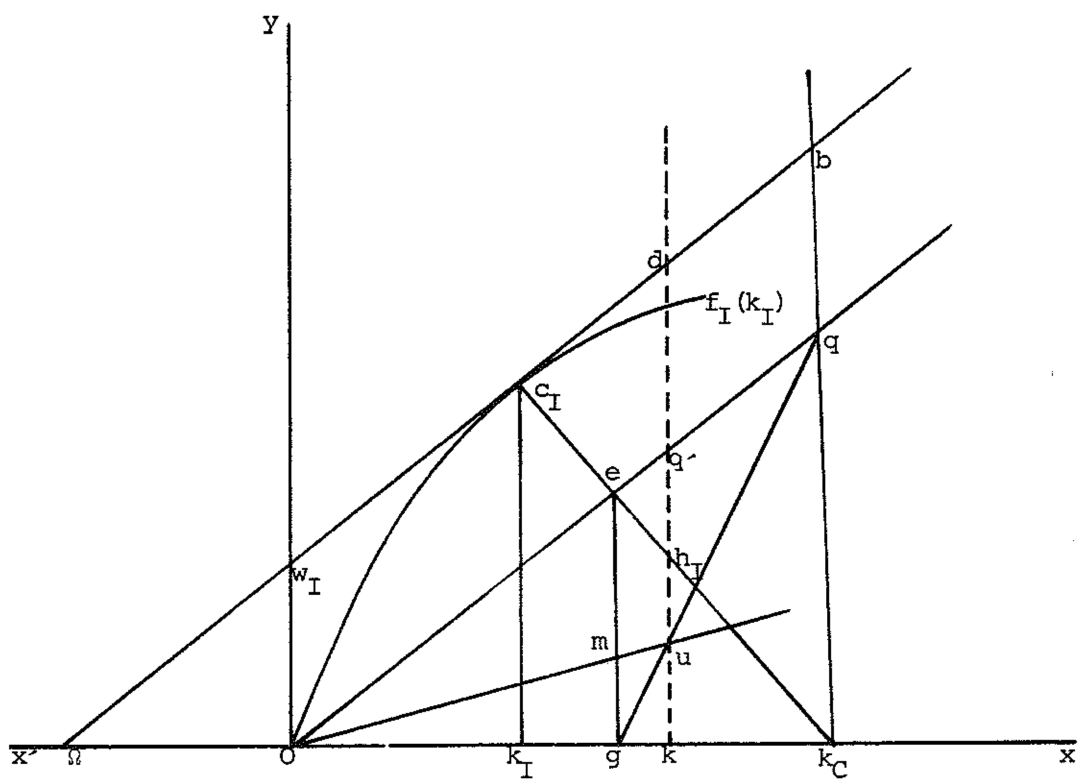

Figure 2

$(\overline{uq}'/\overline{kq}') = (\overline{me}/\overline{ge}) = s_r$. Further, it is obvious that $\overline{h_I q}' = \overline{ku}$, hence $\overline{uq}' = \overline{kh_I}$. In short, one obtains:

$$y_I \equiv \overline{kh_I} = \overline{uq}' \equiv f_I'(k_I) \cdot s_r \cdot k,$$

i.e., equation (4) is satisfied. (Note that the first equality follows from earlier discussion, equation (1).)

It is readily seen that when $s_r = 1$, point m coincides with point g. Since the intersection of lines om and gq is now at point g, point k coincides with point g. This is exactly the way point k was found in Figure 1.

III. The Determination of Long-run Equilibrium

It may be recalled that, in his first article on a two-sector growth model, Professor Uzawa indicated that the uniqueness and stability of the balanced capital–labor ratio are assured if the consumption-goods sector is more capital-intensive than the investment-goods sector. It is somewhat difficult to give economic justifications for this assumption on capital intensity. Therefore, many attempts have been made, notably Drandakis (1963), to obtain alternative conditions for uniqueness and stability for the Uzawa two-sector model. These conditions are briefly summarized in Stiglitz and Uzawa (1969, pp. 406–7) and discussed in Burmeister and Dobell (1970, pp. 120–5). It is, of course, not possible to show rigorously those stability conditions of equilibrium in terms of the diagrams presented in the preceding section. Nevertheless, the following discussion attempts to graphically demonstrate one specific situation in which a condition for stability is satisfied.

Let us first be reminded that the fundamental dynamic equation is:[5]

$$\dot{k} = y_I - (n + \delta)\, k \qquad (5)$$

where n is the constant growth rate of labor forces and δ is the constant depreciation rate of capital goods.

Then, a balanced capital–labor ratio k^* is defined when $\dot{k} = 0$, or $y_I = (n + \delta)k^*$. To show that the growth path described in (5) is stable would mean that for a capital–labor ratio $k < k^*$, $\dot{k}$ should be positive and for a ratio $k > k^*$, $\dot{k}$ should be negative.

With these in mind, we now consider the case where $s_w = 0$, $s_r = 1$, *and* $k_C > k_I$. This represents the extreme form of the case where $s_r > s_w$ which, together with the assumption that $k_C > k_I$, would satisfy a *sufficient* condition for stability.

For given levels of n and δ, one can find a ratio k_I^* such that $f_I'(k_I^*) = n + \delta$. In Figure 3, as soon as k_I^* is determined, point Ω^* is also determined, and so is the level k_C^*. Using the appropriate graphical technique discussed above, one can then easily find k^* in the diagram. It is obvious that $y_I = (n + \delta)k^*$, hence k^* is indeed a balanced capital–labor ratio. Furthermore, due to the neo-classical assumptions for the production functions in both sectors, it is clear that there always exists such a ratio k^* and since only one level of k^* can be found, that k^* is uniquely determined. Suppose now the economy is in a short-run equilibrium at the wage–rental ratio ω and the corresponding capital–labor ratio k. Will this ratio k asymptotically approach the balanced capital–labor ratio k^*?

In Figure 3, ω is selected so that $\omega < \omega^*$ (i.e. $\overline{o\Omega} < \overline{o\Omega^*}$), and obviously, $k_I < k_I^*$ and $k_C < k_C^*$. Also, by noting that as k_I decreases, marginal productivitiy of capital increases, and, by recalling that $y_I = f_I'(k_I) \cdot k$, we have:

$$y_I = f_I'(k_I) \cdot k \equiv \overline{kh_I} > \overline{kv} \equiv (n + \delta) \cdot k;$$

or

$$y_I > (n + \delta) \cdot k\,. \qquad (6)$$

From (5) and (6), it can be seen that k is

[5]Assuming that $(\dot{L}/L) = n$, $\dot{K} = Y_I - \delta K$, and by definition, $k = K/L$, we have: $\dot{k} = (L\dot{K} - K\dot{L})/L^2 = (\dot{K}/L) - k(\dot{L}/L) = y_I - (n + \delta)k$.

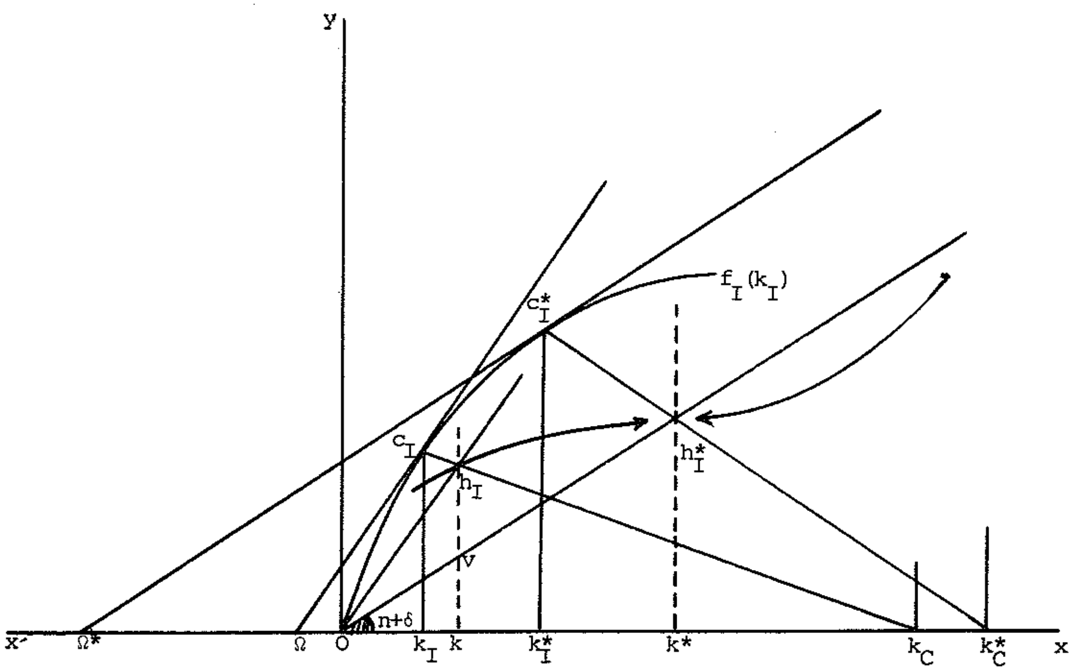

Figure 3

positive under this situation. Hence capital–labor ratio k will continue to increase and this process will be repeated as k asymptotically approaches the uniquely-determined k^*.

It is an easy exercise to see that had we started with a ratio $k > k^*$ and hence, $\omega > \omega^*$, then capital–labor ratio k would decrease and again it would approach k^* as time t tends to infinity. In short, the system is globally stable in this particular example. The arrows in Fig. 3 depict typical paths for ratio k.

IV. Concluding Remarks

One can readily verify several results concerning the comparative-static characteristics of the model from the above graphical discussion. For example, the present technique would enable one to easily identify the signs of partial derivatives of y_I and y_C with respect to a change in the over-all capital–labor ratio k. Furthermore, the direction of impacts of a change in the relative price p, *ceteris paribus,* on other variables can be simply obtained. However, these are left to the interested reader.

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