Utility analysis of chain-weight index
Student Instruction Sheet

Introduction

In this exercise, we will seek to understand why the traditional Laspeyres and Paasche fixed-weight quantity indexes misstate the true growth rate of real GDP, and the Fisher ideal, or chain-weight measure appears to be a good compromise. We will then use the Goal Seek tool of Excel to show that the chain-weight measure is a good approximation of an ideal quantity index that exactly takes into account the change in the cost of living. We assume throughout that in this short-run economy, purchases are made by a representative consumer, and production meets demand.

Suppose an individual representing the economy has Cobb-Douglas preferences represented by the utility function:

$$U = x^a y^b$$

where a and b are positive constants. For an individual facing fixed prices (p_x and p_y) and a fixed income (I), the utility maximizing quantities x^* and y^* are given by the formulas:

$$x^* = \left(\frac{a}{a+b}\right)\frac{I}{p_x} \quad \text{and} \quad y^* = \left(\frac{b}{a+b}\right)\frac{I}{p_y}$$

Suppose initially $a = b = 1$ and that $p_x = \$4$, $p_y = \$1$, and $I = \$400$. Note that the relative price of good x to good y is 4.

We will be using this information to examine how Laspeyres and Paasche fixed-weight quantity indexes (Y^t) measure a change in aggregate quantity over time. The fixed-weight relative quantity index is:

$$Y^t = \frac{p_x^{base} x^t + p_y^{base} y^t}{p_x^{base} x^{base} + p_y^{base} y^{base}}$$

Where the superscript refers to the time period, which is either any given year (t), or the base year (*base*). This is a relative quantity index, where real GDP is shown as a ratio to base year GDP. This is a useful concept because the growth rate of GDP is the relative index minus 1.

Recall that when the base year is in the past, the fixed-weight quantity index is called a Laspeyres index, and when the base year is in the future, the index is called a Paasche index. For example, suppose the base year is 1996. Real (fixed-weight) GDP for the year 1995 would use 1995's quantities and 1996's prices, and so would be a Laspeyres index. Real (fixed-weight) GDP for the year 1997 would use 1997's quantities and 1996's prices, and so would be a Laspeyres index. Because real (fixed-weight) GDP for 1996 uses the prices at the end of 1996, it is considered a Paasche index.

The Fisher ideal, or chain-weight index is a geometric average of the Laspeyres and Paasche indexes, where the chosen base years are the previous year and the current year, where prices are measured at the end of the year. In this case, for any year t, the Laspeyres index for the growth rate of GDP ($\dot{Y}_L^t$) is:

$$\dot{Y}_L^t = \frac{p_x^{t-1} x^t + p_y^{t-1} y^t}{p_x^{t-1} x^{t-1} + p_y^{t-1} y^{t-1}} - 1$$

and the corresponding Paasche index ($\dot{Y}_P^t$) is:

$$\dot{Y}_P^t = \frac{p_x^t x^t + p_y^t y^t}{p_x^t x^{t-1} + p_y^t y^{t-1}} - 1$$

The Fisher ideal, or chain-weight index growth rate ($\dot{Y}_F^t$) is the geometric average of the Laspeyres and Paasche indexes:

$$\dot{Y}_F^t = \sqrt{\dot{Y}_L^t \cdot \dot{Y}_P^t}$$

We are using the geometric average because it is the mathematically correct way to average figures in percentage change terms. In addition, price indexes that use geometric averages have some good properties; in particular time reversibility. This means that for any given price change from the base year to another given year, the price change from the given year back to the base year is the reciprocal of the original price change. Remember for any square root calculation, there are positive and negative roots; make sure to choose the correct sign based on the $\dot{Y}_L^t$ and $\dot{Y}_P^t$ results.

Instructions

1. Set up a spreadsheet to contain the information above. Use the first column (A) to list the parameter and variable names (a, b, p_x, p_y, I, x^*, y^*, U), and the second column (B) to contain the associated values and formulas. (If you don't know how to do italics or subscripts, just use regular formatting.) You may wish to start on row 2, and use row 1 as a label row. (Label the values and formulas as "Original values" in cell B1.) When typing in values, for a, b, p_x, p_y, and I, use the values above. For x^*, y^*, and U, type in formulas based on the appropriate equations above. Remember to start your formulas with an equal sign, use parenthesis to differentiate between the numerator and the denominator of the fractions (as well as different fractions), use an asterisk (*) for multiplication, and use a carrot (^) for exponents. For example, your equation for x^* should be: =(B2/(B2+B3))*B6/B4 If you have set up the problem correctly, you should get $x^* = 50$, $y^*=200$, and $U=10{,}000$.
2. Assume that a year passes, and there has been inflation such that the price of good x rises to \$4.20 and the price of good y rises to \$1.40. (Note that the relative price has fallen to 3, as the price of good y has risen faster than the price of good x.) We want to show what this price change will do to the equilibrium consumption values, initially assuming that income stays the same. Copy the parameter values you just computed (cells B2-B9) to column D (cells D2-D9) of the spreadsheet, so the rows line up as before. Label this new block of cells "New values" in row 1 (cell D1). In this new values block, change the prices to the new values. Note what happens to the quantity demanded of each good, and then note how these new consumption levels affect utility. We are assuming that production equals demand, so these new demand levels are also the new production levels.
3. We now want to calculate the Laspeyres quantity index for the growth rate of GDP. A couple of rows below the "New values" block of cells (cell D12), type in the formula for the Laspeyres quantity index growth rate based on the $\dot{Y}_L^t$ equation above. Use the cell addresses for p_x, p_y, x^*, and y^* in your formula. Again, be careful to put parentheses around the numerator and the denominator, and use the * for multiplication. Use the "Original values" as the base year values, and the new values as the current (t) year. If you typed in the formula correctly, the cell should read –16.7%. [Click the "%" button (next to the "\$" button) in the standard Excel toolbar to format the cell as a percentage. To adjust the number of decimal places, click on the "Increase Decimal" or "Decrease Decimal" buttons (next to the comma button) in the standard Excel toolbar.] So that you can keep track of the information that you are creating, type the phrase "Laspeyres quantity growth rate" in cell C11 to the left of this calculation. [Format the cell for right justify by clicking the appropriate toolbar button.]
4. We now want to calculate the corresponding Paasche quantity index for the growth rate of GDP. In the cell below the Laspeyres figure (D12), type in the formula for the Paasche quantity index growth rate based on the $\dot{Y}_P^t$ equation above. Again use the cell addresses for p_x, p_y, x^*, and y^* in your formula. If you typed in the formula correctly, the cell should read –18.4%. Type the phrase "Paasche quantity growth rate" in cell C11 to the left of this calculation. Note that the Laspeyres and Paasche methods give different answers; in particular, the measured fall in GDP is smaller for the Laspeyres index. To understand why, first note that the good whose quantity fell the most in percentage terms (good y) is also the good whose prices rose by the most in percentage terms. The Laspeyres price index uses the original prices in its calculations when the price ratio of p_x to p_y is highest – equal to four. That is, the price of good y is smallest relative to good x at the original prices, so the relative importance of good y in making up the quantity index is small. The consequence is the impact of the large fall in the production of good y is minimized in the GDP calculation. The Paasche index shows a larger fall in quantity because the price ratio in the "new

values" case is closer (p_x/p_y=3), so good y has the largest possible impact in the quantity index calculation. In summary, the Laspeyres index gives results that are not extreme enough, and the Paasche index gives results that are too extreme. This is a "substitution bias" because it results from consumers substituting away from the good whose relative price rises.

5. A reasonable compromise between the Laspeyres and Paasche indexes is to average them together. This is the Fisher ideal, or chain-weight index. As noted above, the appropriate way to average two growth rates is geometrically. In the cell below the Paasche calculation (D13), write a formula based on the $\dot{Y}_F^t$ equation above using Excel's SQRT (square root) function. Excel's SQRT automatically displays the positive square root, so you will have to start your formula with a minus sign to force the result to be negative. If you entered the formula correctly, the cell should display –17.5%
6. Now we want to find a truly "ideal" quantity index. One way to define a true ideal quantity index (in percentage terms) is the percentage of income taken away from a consumer at initial conditions necessary to leave the consumer at the same level of utility after the price change. That is, it is the change in income that is equivalent to the change in utility. Intuitively, it is a measure of the change in real income because it is the percentage of base year income lost when prices rise, after accounting for substitution effects in changes in demand. To find an exact ideal quantity index for this consumer/economy, we will use the "Solver" tool. Copy the "Original values" cells (column B) to column E, just to the right of the "New values" block of cells. Label this block of cells "Use original values to find ideal quantity index" in row 1.
7. We want to find the level of income that will give the consumer the same utility as after the inflation (6,802.72 units), but at the base year prices. To do this, click on the utility value cell in this new block (which should read 10,000), then click on the Tools menu, then choose "Solver…". A window should appear with entries: "Set cell", "Equal to" and "By changing cells." This is fairly self-explanatory; next to "Set target cell", the utility function cell address should appear (because you clicked on it before starting Solver). Next to "Equal to", select the "Value of" option and type in 6802.72. Next to "By changing cells", type in the address of the income cell (E6) [alternatively, you can click on the income cell; if the cell is blocked by the Goal Seek window, click on the name of the Goal Seek window and drag the window to a new location]. Click OK. You should see the values of I, x^*, y^*, and U change to correspond to U = 6802.72. If you have done this correctly, you should see I = \$329.91, x^* = 41.24, and y^* = 164.96. (Note: Excel may round to more or fewer digits on your computer. To change the number of digits, click on the cell, then on the "Increase Decimal" or "Decrease Decimal" buttons next to the comma button.) These values correspond to the demand for x and y at an income level of \$329.91, which leaves the consumer at a utility level of 6,802.72.
8. We are ready to compute an ideal quantity index growth rate. This is simply the percent difference in income between the new and original prices assuming that utility level is constant – that is the difference in the income (I) levels in columns D and E. Write a formula to do this calculation in cell F11. Be sure to write an "Ideal price index" label for this calculation (in cell E11). If you have done the calculation correctly, you should get an answer of –17.5%. Note that this price index is the same as the chain-weight price index computed earlier. This suggests that the chain-weight index is more than a logical compromise between the Laspeyres and Paasche indexes; it is very close to the true ideal.
9. (optional) An alternative formulation of an ideal price index is the extra income given to the consumer at the new price conditions necessary to return him to the original level of utility. Copy the "New values" cells (column B) to column E, just to the right of the "New values" block of cells. Label this block of cells "Use original values to find ideal quantity index" in row 1.
10. We want to find the level of income that will give the consumer the same utility as before the inflation (10,000 units), at the new prices. To do this, click on the utility value cell in this new block (which should read 6802.72), then click on the Tools menu, then choose "Solver…". A window should appear with entries: "Set cell", "Equal to" and "By changing cells." This is fairly self-explanatory; next to "Set target cell", the utility function cell address should appear (because you clicked on it before starting Solver). Next to "Equal to", select the "Value of" option and type in 10000. Next to "By changing cells", type in the address of the income cell (E6) [alternatively, you can click on the income cell; if the cell is blocked by the Goal Seek window, click on the name of the Goal Seek window and drag the window to a new location]. Click OK. You should see the values of I, x^*, y^*, and U change to correspond to U = 10000. If you have done this correctly, you should see I = \$484.97, x^* = 57.74, and y^* = 173.21.

11. We can now calculate the ideal index quantity growth rate. This is again the percent difference in income between the new and original prices assuming that utility level is constant – that is the difference in the income (*I*) levels in columns D and E, except now as a percent of the adjusted income calculated in the previous step. Write a formula to do this calculation in cell F11. Be sure to write an "Ideal price index" label for this calculation (in cell E11). If you have done the calculation correctly, you should again get an answer of –17.5%.

Notes to instructor

- This sheet uses the terms Laspeyres and Paasche indexes. If students will be confused by this terminology, use terms something like "base year is previous year" and "base year is current year" instead.
- If the growth rate indexes are difficult, it is possible to first calculate "Laspeyres" and "Paasche" real GDP levels for each year, and use these figures to calculate growth rates. Of course, nominal GDP is equal to the exogenous income (*I*) in each period. However, beware that chain weight GDP levels are imputed from calculated chain-weight growth rates, and are not the average of Laspeyres and Paasche levels. (This is not possible because the Paasche and Laspeyres levels use different base years.) To see how to calculate chain-weight levels of GDP, see the next exercise, which is associated with Figure 4.
- An example of a chain-weight price index may also be constructed. This would eliminate the substitution bias associated with the Consumer Price Index. The CPI is always a Laspeyres index. The bias associated with the CPI is explored in Cahill and Kosicki, (*Southern Economic Journal* January 2000).
- Steps 9-11 are shown on the Figure 2 (alternative) ply.
- In steps 6-11, the instructor may prefer to analytically solve for the level of income that meets the utility restrictions, and input the formula into the spreadsheet rather than use Solver.