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On One Parameter Functional Forms for Lorenz Curves

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I. INTRODUCTION

The extensive literature on the measurement of income inequality, the growing literature on global measures of tax progressivity, and recent studies of the combined effect of taxes and governmental benefits on the distribution of income all make use of Lorenz curves. Atkinson (1983) provides extensive citations to the literature on income inequality. Kiefer (1984) cites most of the relevant literature on tax progressivity. A recent study by Lambert (1985) investigates the net redistributive effect of government. Specifying a functional form for the Lorenz curve that is convenient to estimate and interpret is therefore an important research objective. Kakwani and Podder (1976) were the first to propose a specific functional form. Rasche et al. (1980) have pointed out that Kakwani and Podder's specification fails to satisfy the properties of a Lorenz curve, i.e., be nonnegative, nondecreasing and convex on the unit interval, with end points (0, 0) and (1, 1).[1] Gupta (1984) has proposed a log-linear functional form that satisfies these properties and is easy to estimate. This paper takes the log-linear specification as a leading special case and investigates the usefulness of the one parameter functional form for Lorenz curves. It is argued that one parameter functional forms are useful only in limited circumstances.

Rarely is a single Lorenz curve of interest. Most investigations of income inequality, global tax progressivity and governmental net redistributions seek to make comparisons across time or space so that two or more Lorenz curves are required. This paper undertakes to show that use of the log linear or other one parameter functional form to rank income distributions in terms of degrees of inequality is equivalent to rankings based on Gini indices of inequality. Similarly, using the log-linear functional form to estimate tax concentration curves and to rank tax systems in terms of their deviations away from proportionality is equivalent to ranking the tax systems in terms of a Lorenz-Gini based global measure of progressivity. The problem with using a one parameter functional form in comparative studies is that, irrespective of the underlying data, the fitted Lorenz curves can never intersect. Comparisons of income inequality may be subject to the fundamental difficulty first identified by Atkinson (1970) but the fitted one parameter Lorenz curves obscure this essential fact. Similarly, one parameter tax concentration curves may suggest unambiguous comparisons of global tax progressivity when intersections invalidate such comparisons. There are circumstances in which the one parameter functional form can safely be specified without fear of Lorenz crossings. These circumstances are identified and their policy relevance discussed below.

ATKINSON'S THEOREM AND THE LORENZ CURVE FUNCTIONAL FORM

Kakwani and Podder propose the following functional form,

$$Q = \lambda P^{\alpha}(\sqrt{2} - P)^{\beta}, \tag{1}$$

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for $\lambda \geq 0$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, where $Q = (x - y)/2$, $P = (x + y)/2$, and x and y are the cumulative proportions of total population and income, respectively. Rasche, et al. show that this function does not satisfy all of the Lorenz curve properties. They propose the function,

(2) $$y = L_2(x; b, c) = [1 - (1 - x)^b]^{1/c},$$

for $0 < b, c \leq 1$. The log-linear form suggested by Gupta is,

(3) $$y = L_1(x; A) = xA^{(x-1)},$$

for $A > 1$. This function satisfies the necessary properties, and is conveniently estimated as,

(4) $$\ln(y/x) = a(x - 1),$$

where $a = \ln A$.

First some basic facts about Lorenz curves need to be reaffirmed. Let Y denote income and assume income has the distribution function $x = F(Y)$. Gastwirth (1971) shows that the Lorenz curve can be written as

(5) $$y = L(x) = \frac{1}{\mu}\int_0^x F^{-1}(u)\,du$$

where μ is mean income. From (5), we have $Y = \mu L'(x) = F^{-1}(x)$, so choosing a functional form for the Lorenz curve amounts to implicitly specifying the functional form of the distribution of income. It can be shown that the functional form $L_1(x; A)$ implies that the density function of income is everywhere decreasing; empirically, this is not an accurate characterization of income distributions. It also follows from (5) that if the Lorenz curve has k parameters, the income distribution will have, at most, k + 1 parameters. Thus, one parameter Lorenz curves imply the income distribution function has at most two parameters.

More importantly, note that the functional form L_1 approaches the identity function as $A \rightarrow 1$. Consequently, in the log-linear form A can easily be interpreted as a measure of inequality and income distributions can be ranked according to the value of A. It is not obvious but comparisons of income distributions in terms of A is equivalent to ranking the distributions by their Gini indices. This is shown as follows. Integrating (3) yields

(6) $$I(A) = \int_0^1 L_1(x; A)\,dx = a^{-2}(e^{-a} + a - 1).$$

The Gini index is then $G(A) = 1 - 2I(A)$, and is a one-to-one function of A. It follows immediately that G and A always rank income distributions identically.

Atkinson's work reveals an important relation between the distribution of income and social welfare. Assuming an additive and strictly concave social welfare function (SWF) Atkinson demonstrates that for two income distributions with non-intersecting Lorenz curves, the Lorenz ranking is equivalent to the social welfare ranking. Dasgupta, Sen, and Starrett (1973) relax a key assumption and extend Atkinson's result to a wider class of SWFs. They show that Lorenz dominance is equivalent to preference by all increasing S-concave SWFs.[2] The implications of this are clear. If Lorenz curves cross, then there exists a S-concave SWF which ranks the income distributions differently than does from the Gini index. Since G and A rank distributions identically, Atkinson's and Dasgupta, Sen, and Starrett's results apply equally to A, an index equivalent to the Gini measure of income inequality. Shorrocks (1983) extends Atkinson's theorem to the case where the means of the distributions are unequal. Shorrocks defines the generalized Lorenz curve by $L^*(x) = \mu L(x)$ (where μ is mean income) and shows that distributions can be ranked unambiguously if, and only if, the generalized Lorenz curves do not intersect. Since functional forms for the Lorenz curve may be easily adapted to the generalized Lorenz curve, our remarks also apply to generalized Lorenz curves.

The log-linear specification of the Lorenz curve is not the only functional form that yields inequality rankings identical to the Gini coefficient. Other examples include the Lorenz curves associated with the Pareto and log-normal distributions. More generally, let $y = L(x; \Theta)$ be a family of Lorenz curves indexed by a single parameter Θ, and let

(7) $$I(\Theta) = \int_0^1 L(x; \Theta)\,dx$$

be a one-to-one function of Θ. Then the Gini index is also a one-to-one function of Θ, and both G and Θ always yield the same ranking of income distributions. The Atkinson/Dasgupta-Sen-Starrett criticism of the Gini index also applies to the parameter Θ.

Now consider the function $L_2(x; b, c)$. The function L_2 approaches the identity function as $(b, c) \rightarrow (1, 1)$. Thus, points in the parameter space northeast of, say, (b_o, c_o) define Lorenz curves that lie completely above $L_2(x; b_o, c_o)$; the income distributions underlying these Lorenz curves are preferred by all S-concave SWFs. For points southwest of (b_o, c_o) the reverse is true. Identifying states of the world with points in the parameter space, a point (b_1, c_1) is preferred to (b_o, c_o) if $b_1 \geq b_o$ and $c_1 \geq c_o$, with at least one strict inequality.

For L_2, the Gini index is $G(b, c) = (b + c)^{-1} B(1/b, 1/c)$, where B denotes the Beta function. A given value of G defines a negatively sloped curve in the parameter space, and the Lorenz curves defined by the points on this curve will intersect. Then it is clear from the values of b and c that the ranking of the underlying income distributions depends on the SWF chosen. This compares to the one-parameter Lorenz curves, where G defines a unique Lorenz curve, and the underlying distributions are (incorrectly) ranked as equally preferred.

It is well known that Lorenz curves for income distributions among nations and across time may cross. The theorems of Atkinson and Dasgupta, Sen and Starrett show that such intersections have important welfare implications. Therefore, use of a functional form which results in fitted Lorenz curves which cannot possibly intersect should be accompanied by a routine check of the underlying data to ensure that there are no Lorenz intersections.

FURTHER IMPLICATIONS AND CONCLUSIONS

The functional form of Lorenz curves is also relevant to the measurement of tax progressivity. Several global measures of progressivity use the Lorenz-Gini methodology and involve estimates of functions that have the same properties as Lorenz curves. This discussion is limited to Musgrave and Thin's (1948) measure of "effective progression," and this work's extension to include governmental benefits.[3] Musgrave and Thin's measure of tax progressivity is based on the difference between pre- and post-tax Lorenz curves. Specifically, effective progression is $E = (1 - Ga)/(1 - Gb)$, where Gb and Ga are the before and after tax Gini coefficients. Pfähler (1983) and Lambert (1984) extend this approach to include the distributional effects of government expenditures. They use an index of net redistributive effect equal to twice the area between the original (before tax and before expenditure, or BTBE) Lorenz curve and the post-fisc (after tax and after expenditure, or ATAE) Lorenz curve.

Musgrave and Thin's effective progression and Lambert's index of net redistribution can be used to make comparisons across time or across governmental units. Since these measures are straightforward applications of the Lorenz-Gini methodology, it is clear that intersections of the Lorenz curves can easily occur. In such cases a SWF can always be found that ranks tax progression or net redistributive effect differently from the Musgrave-Thin or Lambert measures. Thus, one parameter functional forms may be used to estimate the BTBE and ATAE Lorenz curves, but the analyst must systematically test for Lorenz intersections.

There is, however, one situation in which the one parameter functional form can be used without the necessity of checking for crossings. Given an existing (single) Lorenz curve, policy analysts are often interested in how changes in taxes and expenditures will redistribute income. Consider a linear income tax of the form $T = [Y(1 - \alpha)]\beta$, where β is a constant tax rate and α is the proportion of income, Y, which is exempt from taxation.[4] From the work of Pfähler (1984), it is clear that changes in α (β), holding β (α) constant results in post tax Lorenz curves that do not cross. Thus, the log linear functional form can be used to simulate the redistributive effects of a linear income tax, and there is no need to check for intersecting Lorenz curves. This is the only situation, without checking for crossings, in which one parameter Lorenz curves appear to have a practical use.

Developing a functional form for the Lorenz curve that is convenient to estimate and interpret is an important research objective. The desire for convenience suggests one parameter functional forms, such as the log linear form proposed by Gupta (1984). However, two related characteristics of one parameter Lorenz curves imply that they are only of limited usefulness. First, estimates of the Lorenz curve parameter will yield rankings of income inequality, tax progressivity, and net redistributive effects that are equivalent to indices based on Gini coefficients. Second, Lorenz curves estimates using one parameter functional forms cannot intersect. In general, the use of the log linear or one parameter functional forms in studies of inequality, tax progressivity, and net redistributive effects must be accompanied by systematic efforts to determine whether the relevant Lorenz curves intersect.

NOTES

1. More specifically, a function $y = L(x)$ is a Lorenz curve if it satisfies (i) $L(0) = 0$, (ii) $L(x) \geq 0$ for $0 \leq x \leq 1$, (iii) $L(1) = 1$, (iv) $L'(x) \geq 0$ for $0 \leq x < 1$, (v) $L''(x) \geq 0$ for $0 \leq x \leq 1$, (vi) $L(x) \leq x$ for $0 \leq x \leq 1$, and (vii) $0 \leq \int_0^1 L(x)\,dx \leq 1/2$. The conditions are not all independent.
2. Let B be a bistochastic matrix (all entries nonnegative, all rows and columns sum to one). A function f is Schur-concave if $f(Bx) \geq f(x)$ for all bistochastic matrices B and n-vectors x. The assumption of Schur-concavity incorporates the ideals of anonymity and preference for equality into the SWF.
3. Related measures of progressivity have been developed by Kakwani (1977) and Suits (1977). Formby, Seaks, and Smith (1981) show that the Kakwani and Suits measures differ by a weighting factor equal to the slope of the Lorenz curve. Our remarks apply equally to both of these measures, as both are based on the Lorenz-Gini approach.
4. Lambert (1984) and Browning and Johnson (1984) have recently emphasized the importance of linear income taxes for policy analysis. Lambert points out that the U.K. income tax is very nearly linear. Browning and Johnson investigate a "demogrant" policy consisting of a linear income tax redistributed via lump sum transfers.

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