The Aggregate Factor-price Frontier in Böhm-Bawerk's Period of Production Capital Model: A Graphical Derivation†

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Introduction

The history of capital theory is replete with fables and paradigms.[1] One simple paradigm is the real capital model of J. B. Clark. Basically, it is explained that as the aggregate capital/labor ratio increased so would the ratio of aggregate factor shares (the aggregate wage rate/rate of interest ratio). Or as the aggregate capital/labor ratio increases the wage rate will rise and the rate of interest will fall relatively.[2]

The direct relation between the aggregate capital/labor ratio and the ratio of aggregate factor shares has also been imputed to Böhm-Bawerk's period of production capital model. Böhm-Bawerk's model has been the subject of graphical explanation elsewhere in articles by Dorfman [2] and Hirshleiffer [7].

The objective of this note is to demonstrate how the factor price frontier may be derived from the period of production capital model espoused by Böhm-Bawerk [1] and later by Wicksell [8].

For easier comprehension, Clark's model will be presented first since it is the most familiar.

I. Clark's Parable

Figure 1 illustrates the mathematical formulations of Clark's model. Quadrants I and IV illustrate the direction wages and the rate of interest move as the capital/labor ratio increases. As the capital/labor ratio increases, the wage rate in quadrant I increases while the rate of interest declines in quadrant IV, as capital is accumulated relative to labor. Quadrant III, using a 45° line, transforms the rate of interest in quadrant IV onto the abscissa axis in quadrant II. Thus the quadrant II coordinates are the wage rate and the rate of interest which will be used to develop Samuelson's factor-price frontier. Starting with any capital/labor ratio (k_1), the wage rate (W_1) and the rate of interest (r_1) are found. These coordinates form one point on the factor-price frontier in quadrant IV. Repeating the process, *ceteris paribus*, will form a locus of points for the factor-price frontier in quadrant IV.

The same results regarding the factor-price frontier can be obtained from the period of pro-

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[1]See [4, p. 251], [5] and [6] for details about the paradigms.

[2]See the Appendix for the mathematics of the Clark model.

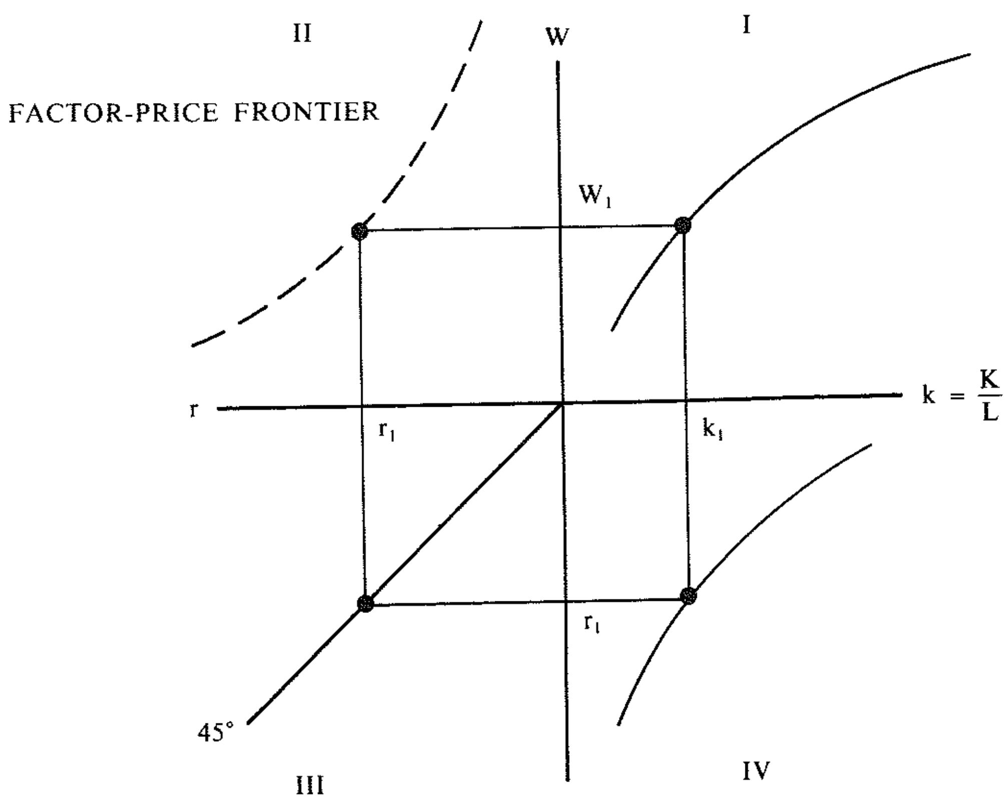

Figure 1 Clark's Parable.

duction model. The following approach is necessary in order to illustrate how both models yield the result that as capital intensity increases factor returns also increase. Ultimately, this relates to the Cambridge Criticism.[3]

II. Interest Rates and the Period of Production: A Prelude

What follows in Figure 2 is a preliminary explanation of the model developed in Figures 3 and 4. The familiar period of production model is reproduced with an additional graphical illustration of interest rates. Several factors are taken as given data, including (1) the value of output per worker function ($W_{(t)}$), which increases at a decreasing rate over time (t, measured on the abscissa), and (2) the wage rate (W_o), which is used to find the maximum simple interest rate (r_o) giving the optimal period of production (t_o). From the tangency of the maximum interest rate and the productivity function, a perpendicular is dropped to the ordinate axis and the distance of this ordinate value to the origin represents the value of output per worker (V_o) for the optimal period of production. Extending the line $w_o r_o$, whose constant slope implies a simple interest rate, through the ordinate axis to a point r on the abscissa axis of the adjacent sector can illustrate the rate of interest. Wicksell showed that the inverse of the distance, or, in the adjacent segment represents the rate of interest [8, pp. 120-126].

[3]In [3, p. xvii] Ferguson identified the Cambridge Criticism and later (p. 258) indicated "the full force of the Cambridge Criticism is as follows: it is illegitimate to make an *a priori* specification of the relation between the aggregate capital-labor ratio and the factor-price ratio; but more important still, it is even illegitimate to make such an *a priori* specification for any *sector* of the economy."

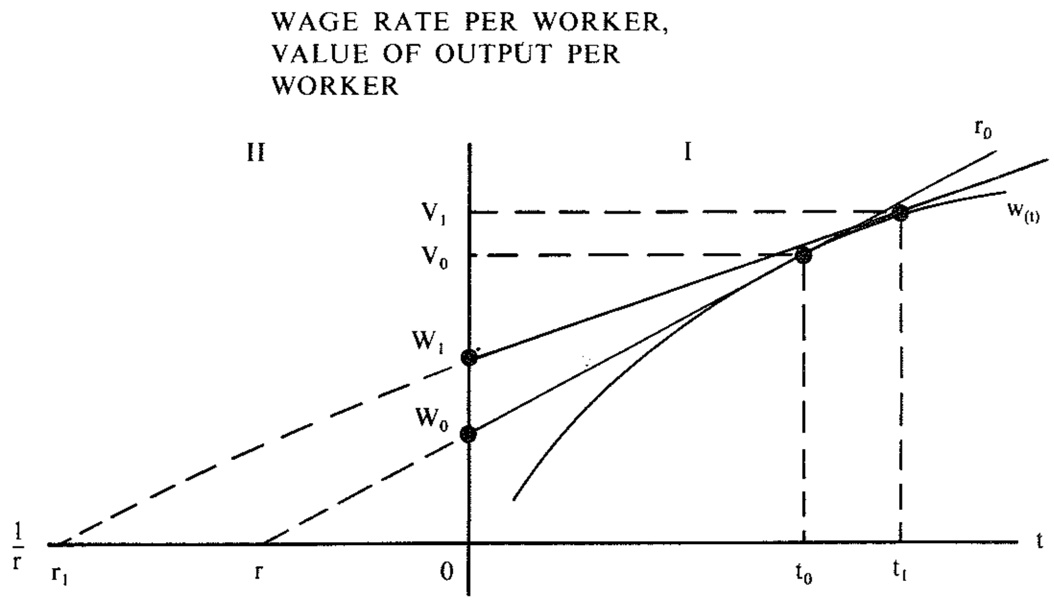

Figure 2 Interest rates and the period of production model: a prelude.

For example, find a new lower rate of interest than the original rate (r_o). If the period of production is extended due to an exogenous increase in capital, *ceteris paribus,* the wage rate rises to W_1, and the maximum rate of interest for the optimal period declines to some r_1 for the period t_1. At the new, longer period, the value of output per worker has risen (to V_1) as capital intensity increases. Since the constant slope of $W_1 r_1$ represents the new, lower interest rate, its extension to the abscissa of the adjacent segment yields an interest rate (r_1) lower than the original rate (r). In other words, the slope of the interest rate line ($W_1 r_1$) is less than the original slope (of $W_o r_o$), and the second rate is less than the original rate of interest.

III. The Period of Production Model and the Factor-Price Frontier

Elements of Figure 2 may be used to develop the factor price frontier in Figure 3. Figure 2 is reproduced as segments I and II of Figure 3. Starting with a wage rate (W_o) the optimal period of production (t_o) is found at the tangency of the interest rate (r_o) with the productivity function ($W_{(t)}$). The rate of interest line ($r_o W_o$) is extended into segment II until it cuts the abscissa (at r).

Now turn to segments III and IV to develop the factor-price frontier. First the 45° line in segment III transforms interest rates from segment II abscissa to the ordinate axis of segment IV. The factor-price frontier is composed of equilibrium wage rates and rates of interest as capital intensity changes, estimated by the length of the period of production; then the wage rate (W_o) must be transformed to segment IV to create one point (with r_o) on the factor-price frontier. A 45° line in segment I will transform the wage rate (W_o) to segment IV's abscissa axis. Thus one point on the factor-price frontier is formed. Repeating the process by altering the wage rate and adjusting the period of production to yield the highest interest rate will, *ceteris paribus,* generate a locus of points for the factor-price frontier.

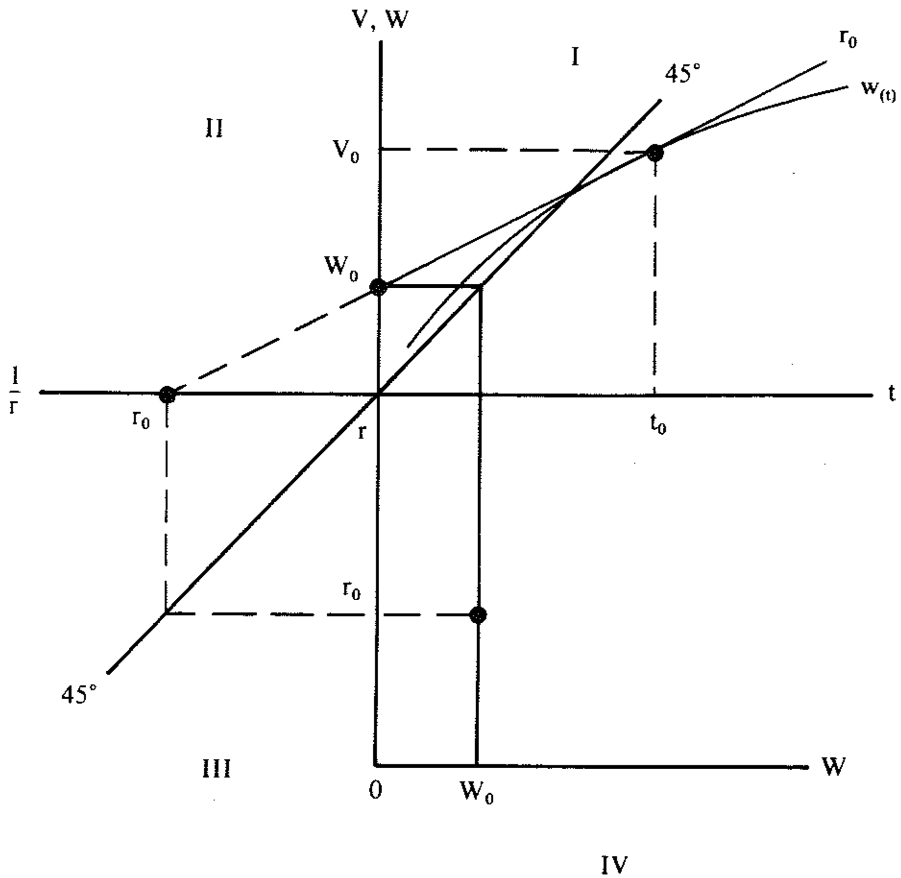

Figure 3 Factor-price frontier and the period of production model.

In a similar manner Figure 4 develops the underlying factor market returns. As capital intensity increases, *ceteris paribus,* the rate of interest declines and the wage rate increases. A simplifying assumption, yielding the special case in Figure 4, is adopted since the graph is quite detailed. It is assumed that an increase in the capital/labor ratio (or capital intensity) results in an equiproportional change in the wage rate so that a series of 45° lines can easily transform wage rates and capital intensity between the five segments of Figure 4. Of course, an implication of this assumption is that relative factor shares are constant. In other words, the factor-price frontier has a constant slope. In fact, it is a mirror image of the relation between the rate of interest and the capital/labor ratio (developed in segment IV).

Figure 4 illustrates the underlying factor returns and capital intensity of the period of production model, but it differs from the preceding figure. Segments I and II are the same as previously shown except that a 45° line has been added in each case. However, the factor-price frontier is developed in segment III instead of segment IV, which segment IV develops the relation between the interest rate and the capital/labor ratio (k). The relation between the wage rate and the capital/labor ratio is illustrated in segment V where it is assumed, for the time being, to be equiproportional.

Begin with the equilibrium conditions of the

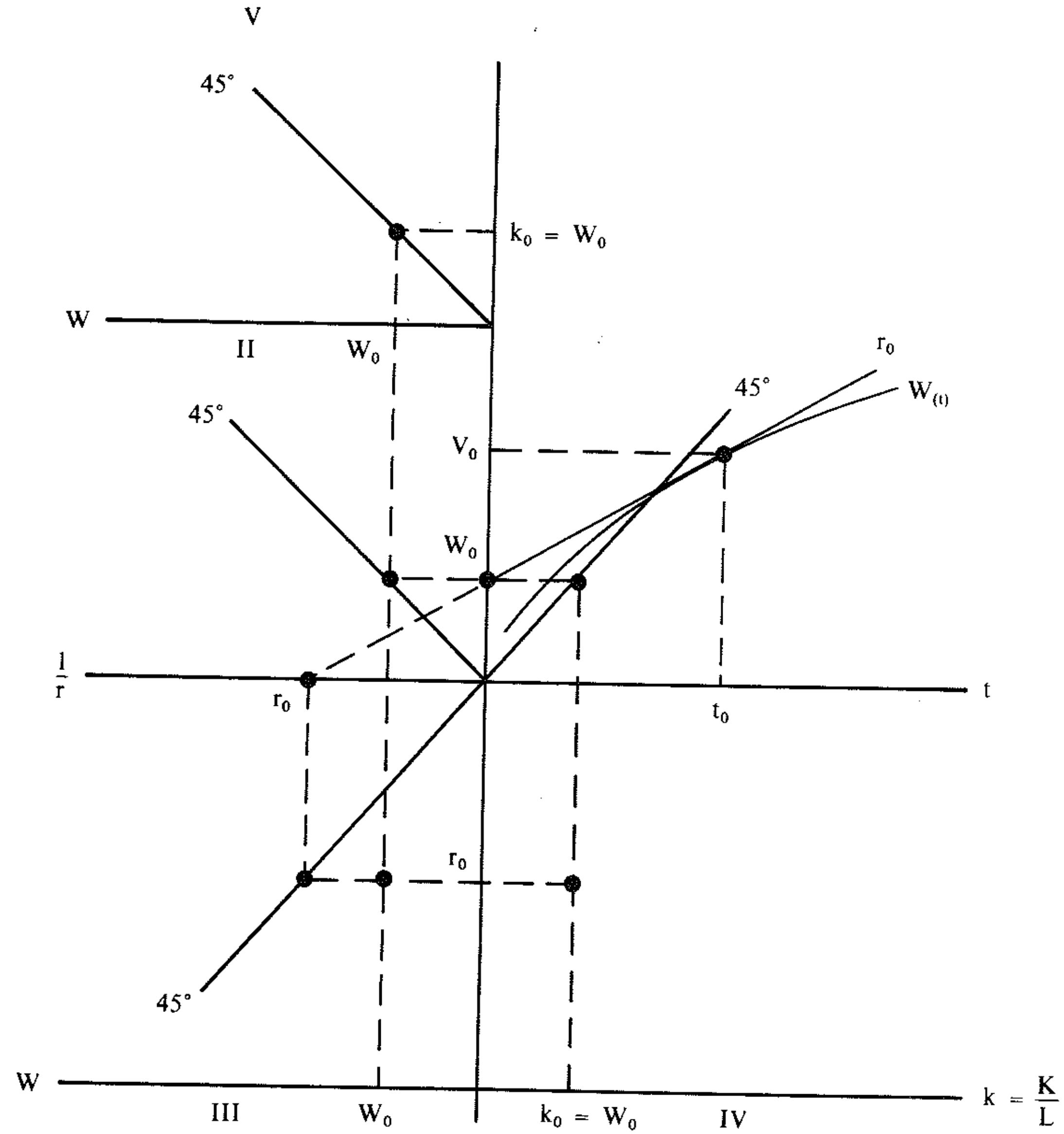

Figure 4 The period of production model and the factor-price frontier with the underlying markets: a special case.

period of production model in segment I. For a given wage rate (W_o), the maximum interest rate (r_o), with respect to the optimal period of production (t_o), is chosen for a given productivity function ($W_{(t)}$). The value of output per capita (V_o) is then read from the optimal period.

Next the rate of interest is transformed into segments III and IV. The interest rate (r_o) from segment II is transformed to the ordinate axis of segments III and IV via the 45° line in segment III. This rate of interest (r_o), on the ordinate axis, is used as one coordinate to locate one point on the factor-price frontier in segment III. It is also used in segment IV to illustrate the relation between capital intensity and the rate of interest (as the former increases the latter decreases).

Then the wage rate (W_o) from segment I is used in segments III, IV, and V to illustrate the factor-price frontier and factor returns. First, the wage rate (W_o) is transformed as the capital/labor ratio in segment IV, via the 45° line in segment I. Since we have assumed an equiproportional relation between the wage rate and the capital/labor ratio, this is sufficient to find the other coordinate (used with r_o) for one point of the rate of interest and capital intensity $\left(k = \frac{K}{L}\right)$. Second, the initial wage rate (W_o) is projected into segment V, via the 45° line in segment II. The assumed proportionality between the wage rate and capital intensity is illustrated by the 45° line in segment V. Finally, the wage rate provides the other coordinate (W_o) for one point on the factor-price frontier. The initial wage rate from segment I is transformed to the abscissa of segment III via the 45° line in segment II. Knowing the interest rate (r_o), one point on the factor-price frontier is found with the wage rate (W_o).

The preceding method can be used to develop the relevant factor-price frontier and the relations between capital intensity $\left(k = \frac{K}{L}\right)$, the wage rate, and the rate of interest. By altering the wage rate, and seeking the optimum period of production with the maximum interest rate, other points, *ceteris paribus*, can be generated. The locus of these points will describe the factor-price frontier and factor returns with respect to capital intensity.

IV. Conclusion

Several observations can be drawn from this exercise. Under the assumption of proportionality between the wage rate and the capital/labor ratio, the following results emerge: (1) as capital intensity increases, the wage rate increases at a constant rate, (2) as capital intensity increases, the rate of interest declines at a constant rate, (3) the factor-price frontier has a constant slope whose Marshallian elasticity yields constant relative factor shares, and (4) the factor-price frontier is a mirror image of segment IV, since the capital/labor ratio equals the wage rate for this example.

The example illustrated in Figure 4 can be generalized by relaxing one assumption. If the equiproportionality assumption between the wage rate and the capital/labor ratio is relaxed, all of the 45° lines are no longer relevant to transform wage rates and rates of interest. The 45° line in segment V would generally increase at a decreasing rate similar to the productivity function. The same new function from V would be needed to transform wage rates into segment IV. Then the factor-price frontier would not reflect constant relative factor shares. And as capital intensity increases the wage rate would increase at a decreasing rate while the rate of interest would decrease.

In conclusion, the preceding graphical approach illustrates the fact that both the real capital and the period of production models yield essentially the same result with reference to an inverse relationship between factor returns. It is this common feature which the Cambridge Critics attack.

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Appendix

The real capital model in Figure 1 can be summarized in mathematical form. Assuming a production function where output depends on capital (K) and labor (L)

$$Q = F(K, L)$$

and the production function is homogeneous of degree one. The function can be rewritten so that the average product of labor (Y) related to the capital-labor ratio (k)

$$Y = F(k)$$

Competitive imputation implies that the wage rate (w) and the rate of interest can be written as:

$$W = F(k) - kf'(k)$$
$$r = F'(k)$$

Thus

$$\frac{dw}{dk} = - kf''(k) > 0$$

Or the wage rate varies directly with the capital-labor ratio and

$$\frac{dr}{dk} = f''(k) < 0$$

Or the rate of interest varies inversely with the capital-labor ratio. Thus

$$\frac{dw}{dr} = \frac{dw}{dk} \div \frac{dr}{dk} = - k < 0$$

Or the wage rate varies inversely with the rate of interest along the factor-price frontier and

$$\frac{r}{w}\frac{dw}{dr} = \frac{r}{w}k = \frac{rK}{rL} \quad \text{ratio of relative shares}$$

Refer to the diagram in Figure 1 and to *The Neoclassical Theory of Production and Distribution* by C. E. Ferguson (Cambridge: Cambridge University, 1969).