STOCHASTIC (s, S) PRICING AND THE U.S. ALUMINUM INDUSTRY

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INTRODUCTION

In some markets, a firm may face certain adjustment costs when it changes nominal price. A mail-order company, for example, incurs costs for printing and mailing new catalogs. Such a firm with market power may not always adjust its nominal price in the face of either inflation or deflation. Instead, the firm may find it optimal to adopt a so-called (s, S) pricing policy under which its nominal price remains fixed as long as the real price remains within a range bounded above by the real price S and below by the real price s. It will only readjust its nominal price when the real price reaches either boundary.

This paper examines two alterations in the traditional (s, S) pricing model. First, price changes are allowed to affect demand rather than costs. In particular demand is expressed as a function of nominal price stability and is directly related to the expected duration of a fixed nominal price.[1] The longer the interval between nominal price changes, the greater the demand. The "adjustment costs" associated with nominal price changes arise via the impact of price variability on demand.

Second, the inflation process is made stochastic. The inflation rate (and, therefore, the rate of change in the real price) at any instant in time has a distribution with some mean and variance, rather than a known, non-stochastic value. This second assumption more accurately mirrors operations in the economy.

A model is developed to predict optimal pricing behavior in these circumstances. Numerical results indicate that when a firm can accurately predict the inflation rate, it is most profitable to keep its nominal price fixed for a period of time. The firm will accept a declining real price in exchange for stimulating demand. On the other hand, when unpredictable inflation causes real prices to vacillate significantly, nominal price stability is not profit maximizing. The firm will revise its nominal price frequently as an erratic inflation rate unpredictably pushes the real price across either boundary. It is more profitable to abandon (s, S) pricing and revise nominal prices in pace with inflation.

The U.S. aluminum industry provides a setting in which to empirically examine the predictions of the model. The demand for aluminum blossomed after World War II, as producers of final goods switched from other metals to aluminum. In Peck's extensive examination of the industry, he wrote that

> transfers [from another metal to aluminum as an input] usually require the modification of either the product or the production process, which, in turn, usually requires capital expenditures. Consequently, the switch from one metal to another is an investment decision entailing the uncertainty and the irreversibility characteristic of these decisions.... [P]rice stability per se increases aluminum consumption by reducing the uncertainties in a purchase with an investment decision. [1961, 22-3]

Thus, the industry satisfies the model's assumption that price variance affects demand rather than costs. Data from this industry are used to determine whether actual pricing policies mirror theoretical predictions.

In the next section, a model is developed to examine optimal pricing policies within the altered (s, S) pricing framework. A numerical technique is then used to find optimal upper and lower (s, S) pricing policy boundaries as a function of the inflation rate. Section three relates our instantaneous mean and variance of the inflation rate. Section four applies the model to pricing in the aluminum industry. We find that empirical results are consistent with theoretical predictions.

THE MODEL

Let $p(t)$ be the nominal price of a commodity produced by a firm with at least some degree of monopolistic power, and let $P(t)$ be the general price index. Then, the real price of the commodity at any time, t, is defined as

$$x(t) = p(t)/P(t). \tag{1}$$

Under an (s, S) pricing policy the nominal price does not adjust instantaneously to changes in the price index. Rather, the nominal price is initially set so that the real price is equal to some optimally determined or exogenously predetermined value, x_o. The nominal price remains fixed until either the real price decays to a lower bound value, s, or climbs to an upper bound value, S. Then the nominal price is revised to restore the real price to x_o.

The optimal price boundaries can either diverge, implying standard (s, S) pricing, or converge, implying instantaneous price-adjustment. Therefore, discrete price-adjustment is neither excluded nor assumed. Rather, solving for the boundaries indicates whether discrete or instantaneous price-adjustment is optimal.

In solving for the optimal upper and lower price boundaries and the initial price the time horizon is infinite and the inflation process is stochastic, displaying a stationary geometric Brownian motion. Assuming a Brownian motion for the inflation process simply means that the rate of change in the real price has a unique distribution associated with it at every instant in time. At any point in time, firms have some expectation about the rate of inflation and the potential expected variation in that rate. (In 1980, for example, people expected high inflation rates and considerable variance. In 1993, they expect a low inflation rate and not much variance.) By convention, the mean and variance of the distribution at each period in time are called the instantaneous mean and instantaneous variance, respectively.

With this in mind, consider firms' expectations about inflation. The expected inflation rate at any time is just the expected mean of the distribution, or the instantaneous mean. Since the economy generally experiences inflation, the real price should fall. The change in the expected real price (or, synonymously, the instantaneous mean), therefore, should be negative. The instantaneous variance describes the likelihood of any deviation between the actual change in the real price and the expected change. Zero variance means the two converge. As the instantaneous variance increases, greater deviations between the expected and the actual inflation rate become increasingly likely.

To characterize the notion that nominal price stability affects demand, we posit the stationary demand function:[2]

$$Q(x(t)) = \alpha - \beta x(t), \tag{2}$$

where $\alpha = \alpha_0[(S - s)/S] + \alpha_1$. The divergence of S from s, relative to S affects the intercept of the demand curve. If there is no nominal price stickiness (so that $s = S$), $\alpha = \alpha_1$. As the nominal price becomes more stable (s and S diverge), α approaches $\alpha_0 + \alpha_1$ and the demand curve shifts upward (which means demand increases for any given real price). Note that for any given set of optimal boundaries, changes in the real price cause movement along a stationary demand curve.[3]

Defining c as constant real average cost per unit of output, instantaneous profits at time t are defined as

$$\pi(x(t)) = [x(t) - c]Q(x(t)). \tag{3}$$

Given the current real price at time zero, the firm's problem is to choose the real price boundaries, S and s, to maximize the total expected discounted profits

$$\Pi(x) = E[\int_0^\infty \pi(x(t))e^{-rt}dt \mid x(0)=x_o], \tag{4}$$

where r is the discount rate. Equation (4) can be converted, following Ye [1984], to a function with the form:[4]

$$\Pi(x) = A(S,s,x_o)e^{\gamma x} + B(S,s,x_o)e^{\lambda x} + \zeta x^2 + mx + n, \tag{5}$$

in which γ, λ, ζ, m, and n are different combinations of parameters in the model, i.e., functions of c, α, β, r, and those characterizing the Brownian motion for the inflation process (the instantaneous mean and variance).

Ideally we would like to take the first-order conditions for maximizing equation (5) with respect to S, s, and x_o, and then solve those equations simultaneously for the optimal values of S, s, and x_o. Given the extreme nonlinearity of the system, this is not possible. However, the optimal S, s, and x_o can be studied numerically.[5]

Using numerical simulation we find that the upper and lower price boundaries concentrate around discrete values (S and s, respectively) and the optimal starting price is near the one-period monopoly price, when the instantaneous variance in the rate of change in the real price is low. As the instantaneous variance increases beyond some point, the boundaries converge. (Note that all three prices are fairly stable until they converge. When (s, S) pricing is finally given up, the nominal price becomes much more erratic.)

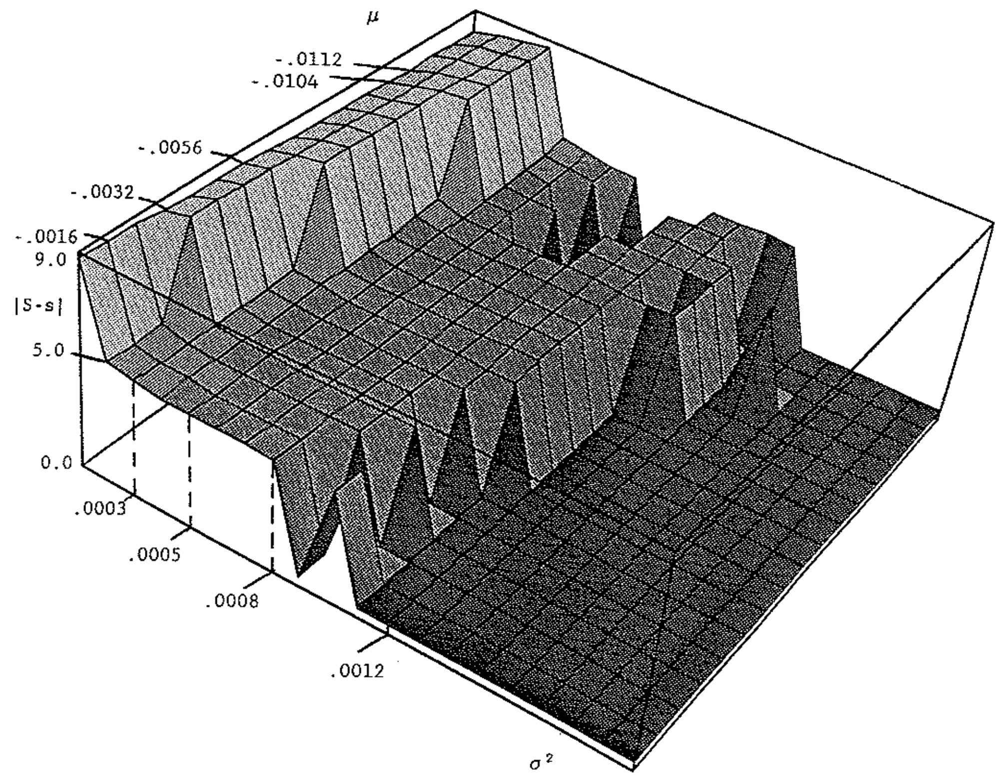
This result is reasonable. It indicates that (s, S) pricing is optimal when the rate of change in the real price exhibits low variance, but eventually is not optimal when the variance gets sufficiently large.[6] Suppose there is no variance in the rate of change in the real price. Then the real price declines smoothly along a known path from the initial price x_0 to the lower bound, s, and is then revised. The optimal strategy for a firm is to set a wide boundary to capture significant demand and to amend its nominal price at predictable intervals. Now consider the case with a high instantaneous variance in the rate of change in the real price. The expected real price should fall. But the actual rate of change can vacillate significantly from the expected value. Hence the real price can unpredictably jump outside the price boundaries at any time, triggering a nominal price revision. These revisions would abate demand. In this situation, the firm finds it optimal to abandon (s, S) pricing altogether.

When a firm can fairly accurately predict the path of its real price over time, it is profit maximizing to engage in (s, S) pricing. Real prices may fall, but the fixed nominal price helps to maintain revenues by stimulating demand. On the other hand, when the real price can unpredictably jump all over the place, the firm may be forced to revise frequently its nominal price (as the price frequently jumps over either the upper or lower boundary). This disrupts demand so much, it ceases to be optimal to maintain an (s, S) pricing policy. Instead, the firm will continually revise its nominal price to keep pace with an oscillating inflation rate.

Numerically we can look at the distance between the boundaries, and by inference, the expected sojourn time. This can be done because the sojourn time is a monotonic function of $(S - s)$. Figure 1 shows the distance between the two price boundaries as a function of the instantaneous mean and variance of the rate of change in the real price. The horizontal axis pointing forward is the instantaneous variance, which increases as it moves away from the origin. The horizontal axis pointing backward is the instantaneous mean, which becomes more negative as it moves away from the origin. The vertical axis is the distance between the two real price boundaries.

For any given instantaneous mean, when the instantaneous variance is near its minimum, the difference between the boundaries and, therefore, the sojourn time, is at its maximum. This distance quickly falls to a plateau as the instantaneous

FIGURE 1
Distance Between Real Price Boundaries as a Function of Instantaneous Mean and Variance of Real Price



variance increases. This plateau is fairly stable for a range of variances. As the instantaneous variance becomes very large, the difference between the boundaries falls to zero. As discussed above, a high instantaneous variance means the rate of change in the real price is less predictable and actual real prices are more likely to fall outside the price boundaries at any moment. Firms are willing to accept some uncertainty, but when the variance gets too high, they abandon (s, S) pricing and adjust prices instantaneously.

For a fixed value of the instantaneous variance, the distance between the two real price boundaries and, therefore, the sojourn time, is generally stable as the instantaneous mean of the rate of change in the real price increases from zero. However, as the instantaneous mean continues to increase, the distance takes a discrete jump upward. This makes sense. The more negative the instantaneous mean, the more rapidly the expected real price reaches the price boundary and triggers a nominal price revision. But frequent revisions act to reduce demand. The firm sets a wider boundary to slow price revisions as the instantaneous mean becomes more negative. Figure 1 also shows that the jump in boundaries occurs at more negative instantaneous mean values as the instantaneous variance increases.

Although our model is different from the standard (*s*, *S*) pricing models, our results are generally consistent with the results in the literature. Sheshinski and Weiss [1977], for example, consider an (*s*, *S*) pricing model that has a fixed cost associated with nominal price adjustments. They show that an increase in the inflation rate (synonymous with the instantaneous mean becoming more negative in our model) increases the distance between the real price boundaries.[7] Our results in Figure 1 are generally consistent with this. Carlson [1989] adjusts the Sheshinski and Weiss model to consider demand elasticity and shows the same qualitative result.

AN EMPIRICAL STUDY OF U.S. ALUMINUM INDUSTRY

In this section, we examine the prediction that the instantaneous mean and variance of the rate of change in the real price affect the optimal price boundaries, and consequently, affect the expected sojourn time. In particular, we test whether the sojourn time is decreasing in the instantaneous variance (controlling for the instantaneous mean). To do this we collected a time series of monthly aluminum prices from 1953 through 1969. Nominal prices from month to month were fairly stable. However, nominal prices displayed discrete jumps at various times.

After deflating the nominal price, the instantaneous mean and variance of the rate of change in the real price is calculated during each period that the nominal price remained fixed. Finally, regression analysis is used to examine, across fixed nominal-price periods, the impact of the instantaneous variance on the sojourn time, while controlling for the instantaneous mean.

Monthly nominal prices for 99.5 percent pure aluminum ingot were obtained from Peck [1961] and various issues of the American Bureau of Metal Statistics *Yearbook*. These monthly nominal prices were divided by their average price from 1947 through 1949 to obtain a monthly nominal price index.[8] Dividing the aluminum nominal price index by the Bureau of Labor Statistics Wholesale Price Index (1947 - 1949 = 100) for non-ferrous metals provides a monthly index of real prices.

The sample contains 19 different fixed nominal price periods. The nominal price began at 20 cents per pound in 1953, rose to 26 cents by late 1957, and fell to 24.7 cents in 1958 and 1959. By 1960, the nominal price rose again to 26 cents per pound and stayed there for 21 months. The nominal price fell in 1961, 1962, and into 1963. By the end of 1963 it began to rise and continued to climb, reaching 27 cents per pound by December, 1969. Sojourn times for these fixed nominal price periods range from 2 months to 24 months with a mean of 9.26 months.

The following regression is used to calculate the instantaneous mean of the rate of change in the real price:

(6) $$ln(x(t_i)) = \phi_i - \mu_i t_i,$$

where i represents a fixed nominal price period, t_i is a month in the fixed nominal price period i, $x(t_i)$ is the real price index value in month t_i, and $-\mu_i$ is the estimated instantaneous mean in period i. Once μ_i is estimated, equation (6) can be used to calculate $E[x(t_i)]$, the expected real index price for each month in fixed nominal price period i. Letting N_i denote the number of months in each fixed nominal-price period, the instantaneous variance of the rate of change in the real price in any period can be estimated from

(7) $$\sigma_i^2 = \Sigma\{x(t_i) - E[x(t_i)]\}^2/N_i.$$

Equation (6) was estimated using Ordinary Least Squares (OLS) and, where necessary, using Generalized Least Squares correcting for first-order autocorrelation.[9] Fifteen of the nineteen estimated instantaneous means were negative (the sign we expected). For those fifteen, eleven were statistically different from zero at the 97.5 percent level using a one-tailed t-test. The others were significant ranging from 81 percent to 95 percent levels using a one-tailed test. The average instantaneous mean was -0.0077. The instantaneous variance was calculated using these results and equation (7). The mean of the estimated instantaneous variances is 0.00037.

To examine the effect of the instantaneous mean and variance on the sojourn time, the following regression is used:

(8) $$\tau_i = \beta_o + \beta_1 VAR_i + \beta_2 MEAN_i VAR_i.$$

In equation (8), τ_i is the sojourn time in period i, and VAR_i and $MEAN_i$ are the estimated instantaneous variance and mean of the rate of change in the real price index in period i, respectively. The interaction term $MEAN_i VAR_i$ allows the mean to alter the variance's impact on the sojourn time. Results from this regression are

(9) $$\tau_i = 7.75 + 32749 VAR_i + 4525834 MEAN_i VAR_i.$$

The adjusted R^2 for the equation is 0.47. The F statistic is 7.1, significant at the 99 percent level. The coefficient β_1 is significant at the 94 percent level, and β_2 is significant at the 90 percent level.

The impact of the instantaneous variance on the sojourn time (while controlling for the mean) can be seen by taking the derivative of τ_i with respect to VAR_i, or

(10) $$\partial\tau_i/\partial VAR_i = 32749 + 4525834 MEAN_i.$$

Evaluating this at the mean value of the instantaneous mean,

(11) $$\partial\tau_i/\partial VAR_i = -2190.$$

For a given instantaneous mean, an increase in the instantaneous variance reduces the sojourn time. This corresponds with our theoretical results as represented in Figure 1. The distance between the real price boundaries, and, by monotonic relationship, the expected sojourn time, decreases as the instantaneous variance increases.

CONCLUSION

The original (*s*, *S*) pricing models were deterministic models which assumed firms pay a fixed dollar-cost when changing nominal prices. Our model extends the literature in two significant ways. First, it transforms the fixed adjustment cost into an impact on demand. The frequency and magnitude of price changes affect the level of demand. In essence, the "cost" of nominal price adjustments is really felt on the revenue side. Second, it considers a stochastic inflation process.

The theoretical results show that the optimal upper real-price boundary, lower real-price boundary, initial price, and length of time that the nominal price is fixed (referred to as the expected sojourn time) all depend on the instantaneous mean and variance of the rate of change in the real price. For a given instantaneous mean, the upper and lower boundaries converge and the expected sojourn time becomes zero when the instantaneous variance gets too large. In other words, if real prices are only somewhat unpredictable, (*s*, *S*) pricing is optimal. When real prices become too unpredictable, (*s*, *S*) pricing is abandoned.

The results also show that for a given instantaneous variance, the boundaries diverge as the instantaneous mean becomes more negative. The more rapidly real prices decay, the wider the boundaries become to stall nominal price revisions.

Data from the U.S. aluminum industry are used for a limited test of the theoretical results. Nominal aluminum prices from month to month over a nearly two-decade period were fairly stable. However, discrete jumps in the nominal price occurred at various times. For each of the fixed nominal-price periods we calculated a sojourn time and the instantaneous mean and variance of the rate of change in the real price. When sojourn times are regressed on the instantaneous means and variances, the regression results correspond to the first theoretical prediction. When the instantaneous mean is controlled, an increase in variance tends to increase the distance between the real price boundaries and the sojourn time.

NOTES

This paper has been presented at the 1989 American Economic Association meetings, Australasian meeting of the Econometric Society, and workshops at the Departments of Economics at Purdue University, Ball State University and the George Washington University. We would like to thank all participants for their comments.

1. Firms may respond to nominal price stability for a variety of reasons. Nominal price stability makes planning easier, both for the supplier and the demander. Frequent nominal price revisions for an input may trigger searches for new input sources by demanders. Nominal input-price stability (and hence a declining real price) may signal the financial viability of an input supplier. This may make potential demanders less reluctant to invest capital into converting plants to use that input.
   A group of new Keynesian economists also have been addressing the issue of sticky nominal prices and the effect on demand: Mankiw [1985]; Akerlof and Yellen [1985]; Ball and Romer [1990]; Calpin [1985]; Calpin and Spulber [1987]; and Carlton [1986]. The issue of real price versus nominal price stickiness is also addressed in Mankiw and Romer [1991].
2. For further details see Rosenbaum and Ye [1989]. Nonstationary demand cases will be dealt with by the authors in a future paper.

3. Consumers are treated as myopic in the sense that they do not expect the impending revision of the nominal price as the real price approaches either boundary. This rules out any impact of future price expectations on current demand.
4. An analytical derivation of solutions to the maximization of equation (4) is available from the authors upon request.
5. A detailed report on numerical studies is available from the authors upon request.
6. Barro [1972] gets a similar result.
7. Sheshinski and Weiss [1977] also explicitly solve for the sojourn time. Since our model is not deterministic, the sojourn time is stochastic and we can not explicitly calculate it. Therefore, results with respect to the sojourn time are not directly comparable.
8. Fisher [1962] defends the use of this deflator.
9. Autocorrelation was a problem (as indicated by a Durbin-Watson statistic below the lower boundary) in 5 of the 19 fixed nominal price periods. The Durbin-Watson statistic fell between the boundaries in a sixth period. However, for that period the first-order autocorrelation coefficient was not statistically different from zero, so the OLS results were used.

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