Output of the Constrained Revenue Maximizing Firm

C. A. KNOX LOVELL*

William J. Baumol (1967) has developed an oligopoly model in which the firm is assumed to seek maximum total revenue, subject to a minimum profit constraint.[1] The model has attracted attention as a potentially fruitful alternative to the traditional unconstrained profit maximization monopoly model, although to date the testable implications of the two models are disappointingly similar.

One well-known implication of Baumol's model asserts that the firm produces a greater rate of output, and reaps a smaller volume of profits, if it is a constrained revenue maximizer (*CRM*) than if it is a profit maximizer (ΠM). This implication has gained wide acceptance, provided the profit constraint takes the form of a minimum dollar value, say \$2 million per annum. However Baumol notes (p. 66) that

> . . . in practice, few businessmen have such explicit and clearcut profit goals. And it may be expected that in many cases the firm's minimum profit goal will be approximated better by the requirement that it earns at least, say, 8 per cent on its investment or 20 per cent on its costs or 15 per cent on its dollar sales.

In light of this expectation it seems worthwhile to ascertain whether or not the posited relationship between rates of output under the two alternative behavioral assumptions survives these three respecifications of the profit constraint. Baumol (pp. 66–68) seems convinced that it does. In general it does not. The relationship between the rate of output of a *CRM* firm and that of a ΠM firm depends crucially upon the form of the profit constraint under which the *CRM* firm operates, a point that has not been recognized in the literature. The purpose of this note is to demonstrate and interpret this dependence.

I. Output Under Alternative Specifications of the Profit Constraint

Consider an oligopolistic firm seeking to maximize revenue subject to a profit constraint that is allowed to assume various forms. Define the following variables:

- $X \geqq 0$ rate of output, in physical units;
- $S \geqq 0$ total selling expense, $S = \Sigma S_i$, where $S_i \geqq 0$ is the expenditure on the i-th type of sales effort;
- $C(X)$ total production cost, where $C'(X) > 0$;
- $R(X, S)$ total revenue, assumed to be a concave function with $\partial R(X, S)/\partial S_i > 0$;
- $\Pi(X, S)$ total profits, $R(X, S) - C(X) - S$;
- $\Pi_1(X, S)$ profit constraint expressed in dollar terms;
- $\Pi_2(X, S)$ $= \Pi_2^* [R(X, S)]$, profit constraint expressed as a per cent (Π_2^*) of total revenue;
- $\Pi_3(X, S)$ $= \Pi_3^* [C(X) + S]$, profit constraint expressed as a per cent (Π_3^*) of total cost;
- $\Pi_4(X, S)$ $= \Pi_4^* [\theta_0 + \theta_1 C(X) + \theta_2 S]$, profit constraint expressed as a rate of return (Π_4^*) on investment, where investment is taken to be a linear function of the two types of cost.

*University of North Carolina.

[1] The profit constraint, whatever its form, is an internally imposed minimum thought necessary to finance future growth and to satisfy stockholders, rather than an externally imposed maximum set by a regulatory agency. The firm is not subject to regulatory constraint.

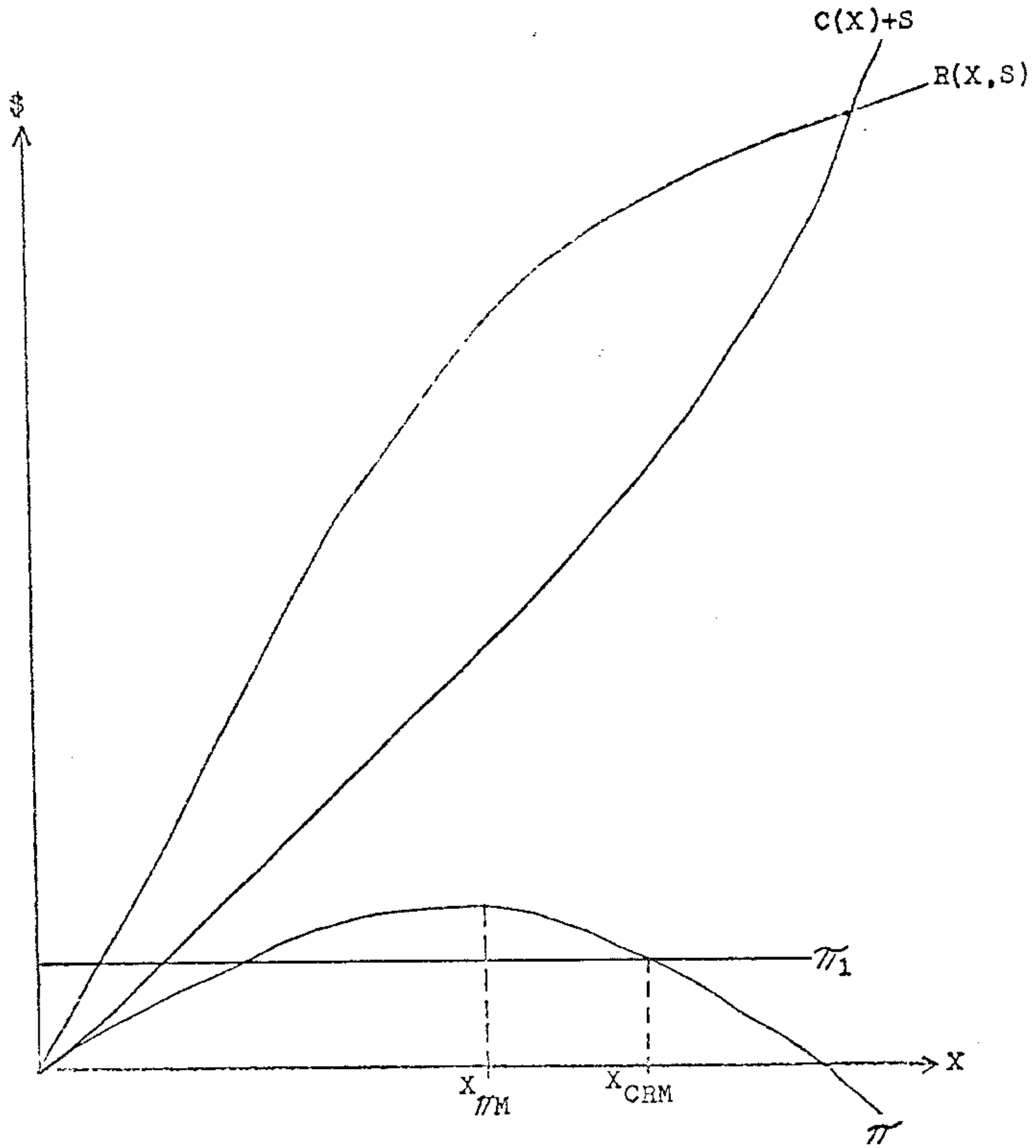

Figure 1

The four specifications of the profit constraint are Baumol's, although he considers only the first in any detail. The assumption that $\partial R(X, S)/\partial S_i > 0$ implies that the profit constraint must be binding in equilibrium regardless of specification. Thus the problem is one of maximizing a concave objective function subject to a concave functional equality constraint and nonnegativity constraints on all variables, and the Kuhn-Tucker conditions are necessary and sufficient for a maximum.

Consider first Baumol's problem of maximizing revenue subject to a profit constraint expressed in dollar terms. Forming the Lagrangean

$$G(X, S, \lambda_1) = R(X, S) + \lambda_1 [R(X, S) - C(X) - S - \Pi_1(X, S)], \qquad (1a)$$

the Kuhn-Tucker conditions are

$$\frac{\partial R}{\partial X} + \lambda_1 \left(\frac{\partial R}{\partial X}\right) - \lambda_1 C'(X) \leqq 0; X \geqq 0 \quad (1b)$$

$$\frac{\partial R}{\partial S_i} + \lambda_1 \left(\frac{\partial R}{\partial S_i}\right) - \lambda_1 \leqq 0; \quad S_i \geqq 0 \qquad (1c)$$

$$R(X, S) - C(X) - S - \Pi_1(X, S) = 0;$$

$$\lambda_1 > 0. \qquad (1d)$$

Using $\lambda_1 > 0$ from (1d) in (1c) gives $\partial R/\partial S_i < 1$. Using $\lambda_1 > 0$ in (1b) gives $\partial R/\partial X = (\lambda_1/1 + \lambda_1)C'(X) < C'(X)$ for $X > 0$. In the ΠM model equilibrium rates of sales effort and output are determined by the conditions $\partial R/\partial S_i = 1$ for all $S_i > 0$ and $\partial R/\partial X = C'(X)$ for $X > 0$.[2] Hence for this specification of the profit constraint sales effort and the rate of output are greater under *CRM* than under ΠM. A comparison of the two rates of output appears in Figure 1.

However if the profit constraint is respecified, the relationship between *CRM* and ΠM rates of sales effort and output may be reversed. For example if the profit constraint is defined as a minimum return on total revenue the Lagrangean becomes

$$G(X, S, \lambda_2) = R(X, S) + \lambda_2 [R(X, S) - C(X) - S - \Pi_2(X, S)], \qquad (2a)$$

with Kuhn-Tucker conditions

$$\frac{\partial R}{\partial X} + \lambda_2 \left(\frac{\partial R}{\partial X}\right) - \lambda_2 C'(X) - \lambda_2 \Pi_2^* \left(\frac{\partial R}{\partial X}\right) \leqq 0;$$

$$X \geqq 0 \qquad (2b)$$

$$\frac{\partial R}{\partial S_i} + \lambda_2 \left(\frac{\partial R}{\partial S_i}\right) - \lambda_2 - \lambda_2 \Pi_2^* \left(\frac{\partial R}{\partial S_i}\right) \leqq 0;$$

$$S_i \geqq 0 \qquad (2c)$$

$$R(X, S) - C(X) - S - \Pi_2(X, S) = 0;$$

$$\lambda_2 > 0. \qquad (2d)$$

Using $\lambda_2 > 0$ from (2d) in (2c) gives $\partial R/\partial S_i \gtreqless 1$ according as $\Pi_2^* \gtreqless 1/\lambda_2$ for all $S_i > 0$. If the target rate of return on total revenue (Π_2^*) is set sufficiently high, sales effort may be so restricted as to force marginal revenues above unity. Using this result in (2b) gives $\partial R/\partial X = (\lambda_2/1 + \lambda_2 - \lambda_2 \Pi_2^*)\, C'(X) \gtreqless C'(X)$ according as $\Pi_2^* \gtreqless 1/\lambda_2$ for $X > 0$. Again if the target rate of return on total revenue is set sufficiently high, the resulting restriction on sales effort leads to a diminution of demand and a reduction of output below the ΠM rate. For Baumol's result to obtain under this specification of the profit constraint it is necessary that the target rate of return on total revenue be low enough to satisfy $\Pi_2^* < 1/\lambda_2$. Both possibilities are illustrated in Figure 2, where Π_{21}^* and Π_{22}^* have been selected in such a way as to yield $X_{CRM1} < X_{\Pi M} < X_{CRM2}$.

If the profit constraint is defined as a minimum return on total cost, the Lagrangean becomes

$$G(X, S, \lambda_3) = R(X, S) + \lambda_3 [R(X, S) - C(X) - S - \Pi_3(X, S)], \qquad (3a)$$

with Kuhn-Tucker conditions

$$\frac{\partial R}{\partial X} + \lambda_3 \left(\frac{\partial R}{\partial X}\right) - \lambda_3 C'(X) - \lambda_3 \Pi_3^* C'(X) \leqq 0;$$

$$X \geqq 0 \qquad (3b)$$

$$\frac{\partial R}{\partial S_i} + \lambda_3 \left(\frac{\partial R}{\partial S_i}\right) - \lambda_3 - \lambda_3 \Pi_3^* \leqq 0; \quad S_i \geqq 0 \qquad (3c)$$

[2] In equilibrium λ_1 represents the marginal revenue of relaxing the functional constraint $\Pi_1(X, S)$, provided there are no effective direct constraints on the adjustment of X and S_i. If there is an effective direct constraint on X and S_i, this limits their adjustment and tends to reduce the marginal revenue of relaxing the functional constraint. The same analysis holds for the three other forms of the profit constraint.

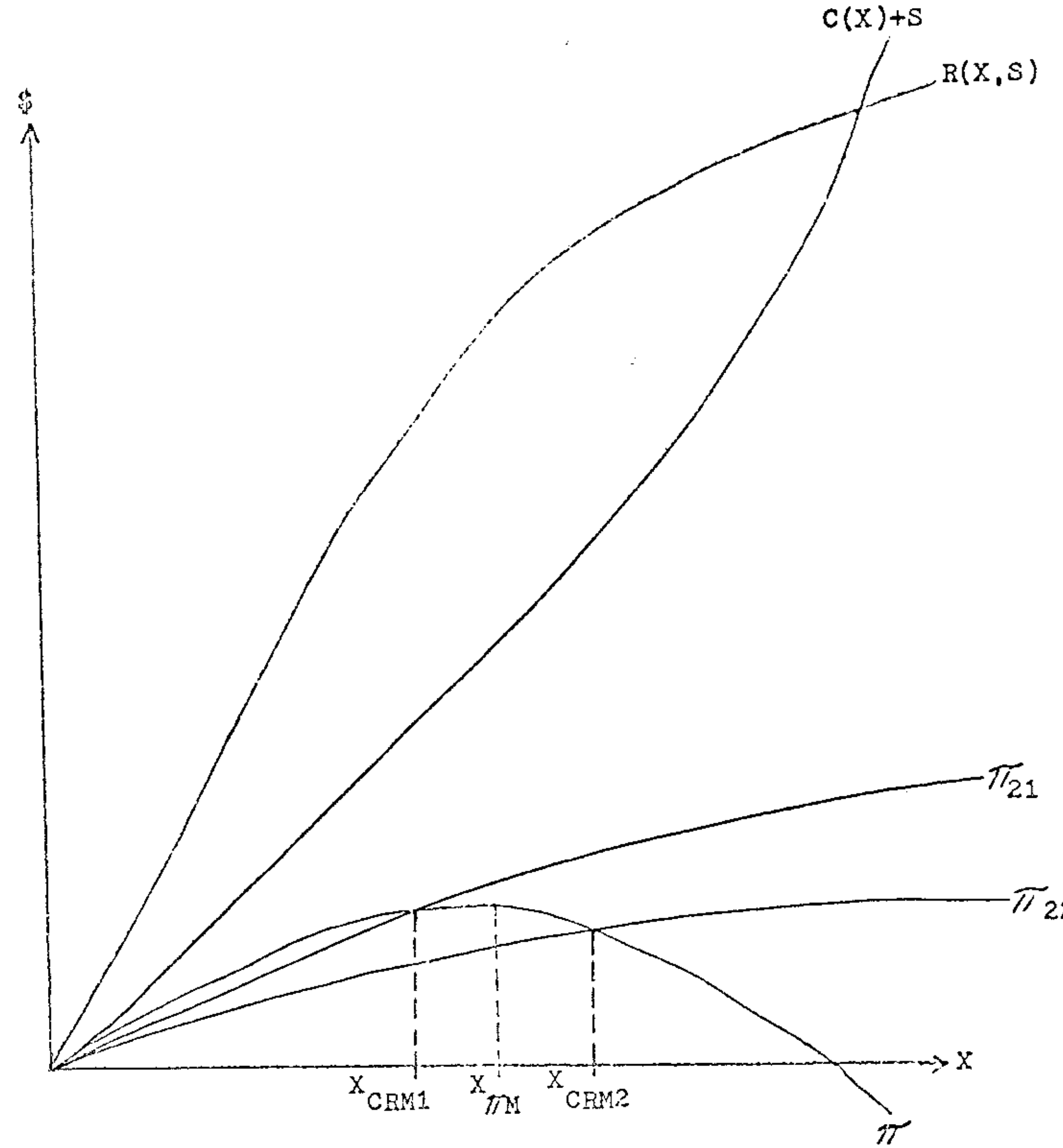

Figure 2

$$R(X, S) - C(X) - S - \Pi_3(X, S) = 0;$$

$$\lambda_3 > 0. \qquad (3d)$$

Using $\lambda_3 > 0$ from (3d) in (3c) gives $\partial R/\partial S_i \gtreqless 1$ according as $\Pi_3^* \gtreqless 1/\lambda_3$ for all $S_i > 0$. Using this result in (3b) gives $\partial R/\partial X = [\lambda_3(1 + \Pi_3^*)/1 + \lambda_3]\, C'(X) \gtreqless C'(X)$ according as $\Pi_3^* \gtreqless 1/\lambda_3$ for $X > 0$. If the target rate of return on total cost (Π_3^*) is set sufficiently high, the *CRM* rate of output can fall short of the ΠM rate. The reasoning is the same as above: a high target rate of return leads to a restriction on sales effort, which in turn causes a reduction in output. This case is illustrated in Figure 3, where

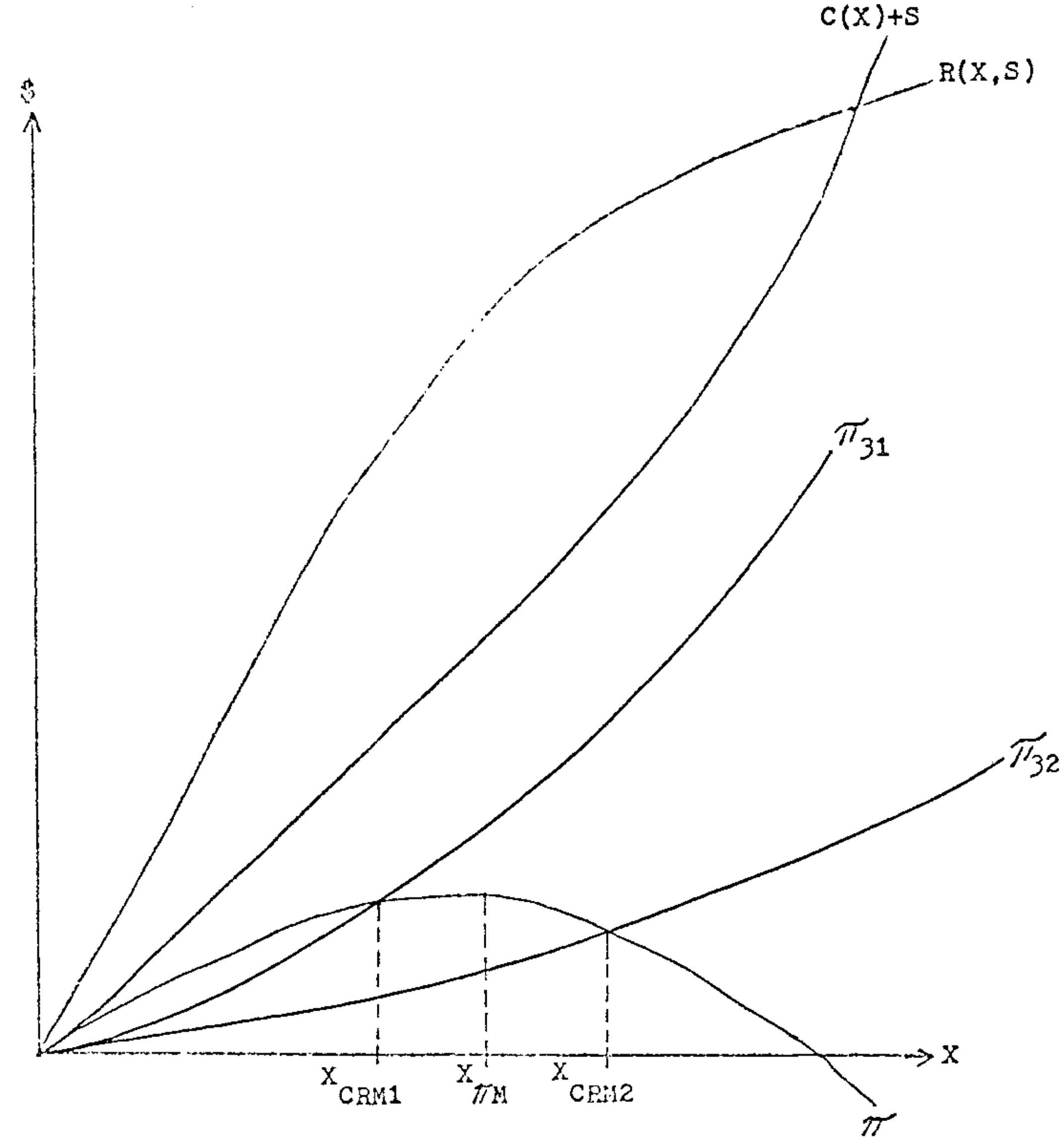

Figure 3

Baumol's result holds only if the target rate of return is low enough to keep $\partial R/\partial S_i < 1$ for all $S_i > 0$.

Finally, if the profit constraint is defined as a minimum return on investment, the Lagrangean becomes

$$G(X, S\lambda_4) = R(X, S) + \lambda_4\,[R(X, S) - C(X) - S - \Pi_4(X, S)]\,, \qquad (4a)$$

with Kuhn-Tucker conditions

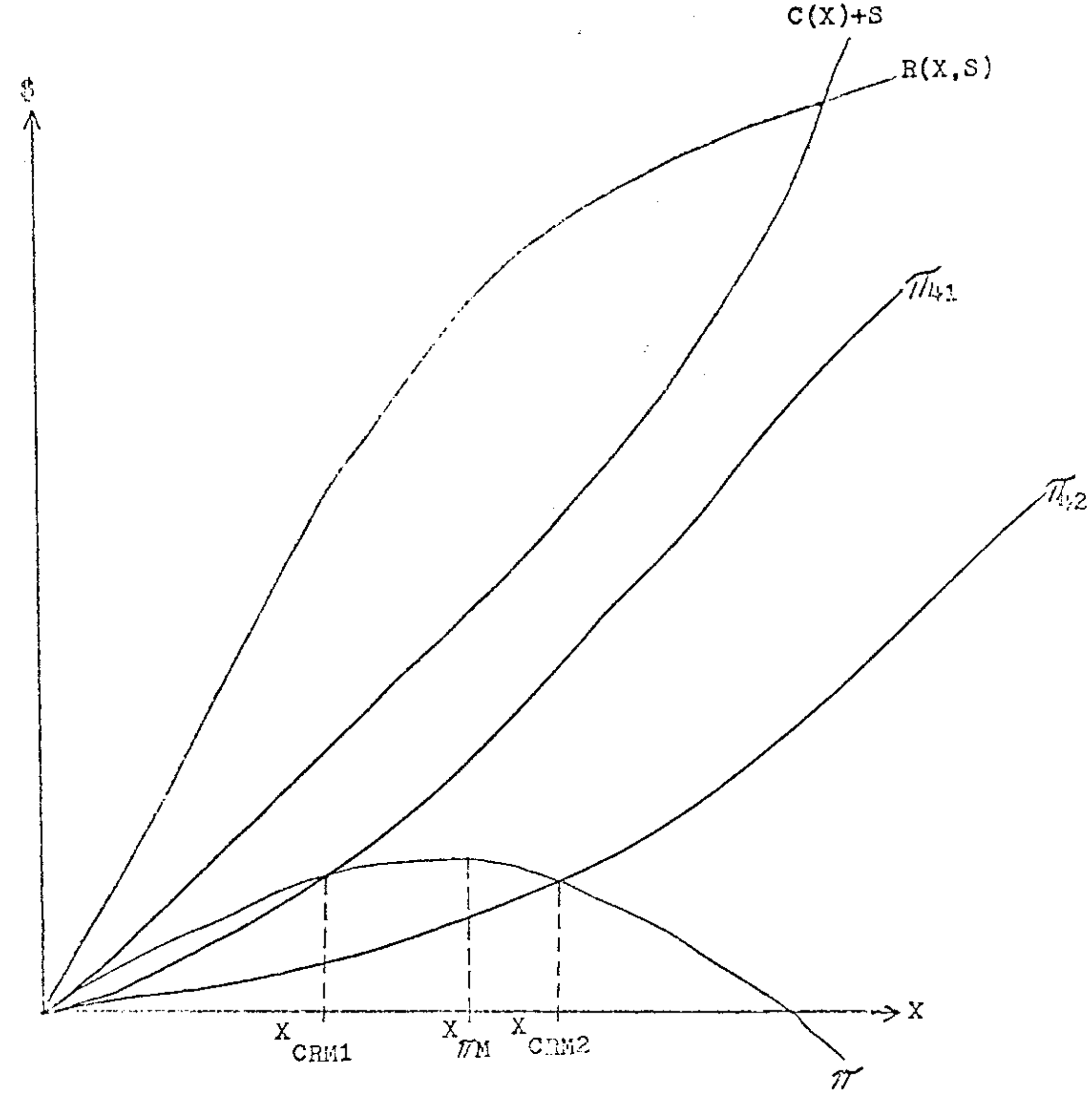

Figure 4

$$\frac{\partial R}{\partial X} + \lambda_4 \left(\frac{\partial R}{\partial X}\right) - \lambda_4 C'(X) - \lambda_4 \Pi_4^* \,\theta_1 C'(X) \leqq 0; \qquad X \gtreqless 0 \qquad (4b)$$

$$\frac{\partial R}{\partial S_i} + \lambda_4 \left(\frac{\partial R}{\partial S_i}\right) - \lambda_4 - \lambda_4 \Pi_4^* \,\theta_2 \leqq 0; \quad S_i \gtreqless 0 \qquad (4c)$$

$$R(X, S) - C(X) - S - \Pi_4(X, S) = 0; \quad \lambda_4 > 0. \qquad (4d)$$

Using $\lambda_4 > 0$ from (4d) in (4c) gives $\partial R/\partial S_i \gtreqless 1$ according as $\Pi_4^* \gtreqless 1/\theta_2\lambda_4$ for all $S_i > 0$. Using this result in (4b) gives $\partial R/\partial X = [\lambda_4(1 + \theta_1 \Pi_4^*)/1 + \lambda_4]\; C'(X) \gtreqless C'(X)$ according as $\Pi_4^* \gtreqless$

$1/\theta_1\lambda_4$ for $X > 0$. In this case the target rate of return on investment (Π_4^*) may be high enough to force a restriction on sales effort sufficient to cause marginal revenues to exceed unity. However in contrast to the second and third cases this is neither necessary nor sufficient to cause a reduction in output beneath the ΠM rate, which requires $\Pi_4^* > 1/\theta_1\lambda_4$. The possible outcomes of this specification of the profit constraint are illustrated in Figure 4.

II. An Interpretation of the Results

One of the principal implications of Baumol's model of unregulated constrained revenue maximization asserts that the firm produces a greater rate of output under *CRM* than under ΠM. It has been demonstrated that this proposition lacks general validity, and that the relationship between rates of output under the two hypotheses depends upon the specification of the profit constraint. Only when the profit constraint takes the form of a minimum dollar value does $X_{CRM} > X_{\Pi M}$. For each of three other specifications of the profit constraint, $X_{CRM} \lesseqgtr X_{\Pi M}$ according as the target rate of return exceeds, equals or falls short of some critical value. This result is of some importance since, as Baumol acknowledges, firms are most likely to adopt one of these three types of constraint in practice.

Finally, it should be noted that this modification of Baumol's analysis has little effect on any further implications of the *CRM* hypothesis. It does alter the tenor of his implications concerning oligopoly and ideal output. But the response of a *CRM* firm to changes in overhead costs or lump sum taxes remains unchanged. And the theorems of Bailey and Malone (1970) and of Baumol and Klevorick (1970) concerning the behavior of the *CRM* firm under rate of return regulation (the reverse A-J effect) remain valid.

References

Bailey, Elizabeth E. and John C. Malone, "Resource Allocation and the Regulated Firm," *The Bell Journal of Economics and Management Science*, I (Spring, 1970), 129–42.

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